AN EXERCISE ON FABER POLYNOMIALS

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Theorem 1. Let $M_0^! = \mathbb{C}[j]$ be a space of weakly holomorphic modular forms of weight 0. $M_0^!$ has a canonical basis $\{j_m(z)\}_{m=0}^{\infty}$, where $j_m(z) = q^{-m} + O(q)$. Then, we have:

$$\sum_{m=0}^{\infty} j_m(\tau)q^m = \frac{E_4^2 E_6}{\Delta}\frac{1}{j(z)-j(\tau)}.$$

Proof. Let $j(z) = q^{-1} + \sum_{m=0}^{\infty} b_m q^m$, and we have $\frac{E_4^2E_6}{\Delta} = -\frac{1}{2\pi i}j'$. Also, we remark that $-\frac{1}{2\pi i}\frac{qj'}{q(j(z)-j(\tau))}$ has a convergent power series for $|q| < r$ for some $r > 0$. We prove the theorem by showing that both sides are equal in the formal power series ring $\mathbb{C}[[q]]$. For, we need:

$$\left(\sum_{m=0}^{\infty} j_m(\tau)q^m\right) q(j(z)-j(\tau)) = -\frac{1}{2\pi i}qj'. \tag{1}$$

After this, we obtain that $\sum_{m=0}^{\infty} j_m(\tau)q^m$ equals to $\left(-\frac{1}{2\pi i}qj'\right)[q(j(z)-j(\tau))]^{-1}$ as an element in $\mathbb{C}[[q]]$, which gives Theorem1. Now, we multiply two power series,

$$\left(\sum_{m=0}^{\infty} j_m(\tau)q^m\right) q(j(z)-j(\tau)) = 1+\sum_{m=1}^{\infty}\left(j_{m+1}(\tau) + \sum_{m_1+m_2=m} j_{m_1}(\tau)b_{m_2} - j_m(\tau)j(\tau)\right) q^{m+1}.$$

We claim that:

$$j_{m+1}(\tau) + \sum_{m_1+m_2=m} j_{m_1}(\tau)b_{m_2} - j_m(\tau)j(\tau) = -mb_m. \tag{2}$$

Recall that j_m is a polynomial of j for each m, and by multiplying q-expansions of the last term, we obtain that the coefficients of negative powers of q in LHS are all 0, thus we have the LHS of (2) is a holomorphic modular form of weight 0, which is a constant. Moreover, (2) is equivalent to $j_m(\tau) = q^{-m} + mb_mq + O(q^2)$.

Let

$$j^m = q^{-m} + \sum_{0\le n\le m-1} b_{m,-n}q^{-n} + b_{m,1}q + O(q^2). \tag{3}$$

Then, we have

$$j_m = j^m - \sum_{0\le n\le m-1} b_{m,-n}j_n. \tag{4}$$

Our claim is obvious when $m = 1$, for $j_1 = j - b_0 = q^{-1} + b_1q + O(q^2)$. Suppose that the claim is true for all $1 \le m' < m$, i.e. $j_{m'}(z) = q^{-m'} + m'b_{m'}q + O(q^2)$ holds for $1 \le m' < m$. Differentiating j with respect to q gives,

$$\frac{d}{dq}j = -q^{-2} + \sum_{n=1}^{\infty} nb_nq^{n-1}. \tag{5}$$

Comparing the coefficient of q^{-1} in the series $j^m \frac{d}{dq} j = \frac{1}{m+1} \frac{d}{dq} j^{m+1}$ gives,

$$mb_m + \sum_{1 \le n \le m-1} nb_{m,-n} b_n - b_{m,1} = 0. \tag{6}$$

In view of (4), we obtain $j_m = q^{-m} + mb_m q + O(q^2)$, which completes the proof of our claim by induction. Now, Theorem1 follows by $-\frac{1}{2\pi i} qj' = 1 + \sum_{m=1}^{\infty} (-mb_m) q^{m+1}$.