1. (a) $xy' - 2y = 2x^4$

homogeneous: $y' - \frac{2}{x} y = 0$

$$y = Ce^{2\ln x} = Cx^2$$

inhomogeneous: $y = C(x)x^2$

$$C'(x)x^2 + \cancel{C(x)2x} - \cancel{C(x)2x} = 2x^3$$

$$C'(x) = 2x$$

$$C(x) = x^2 + C$$

$$y = x^4 + Cx^2.$$

(b) $y' + y\tan x = \sec x$

homogeneous: $y' + y\tan x = 0$

$$\ln|y| = -\int \tan x\,dx = \int \frac{d\cos x}{\cos x} = \ln|\cos x| + C$$

$$y = C\cos x$$

inhomogeneous:

$$C'(x)\cos x = \frac{1}{\cos x}$$

$$C'(x) = \frac{1}{\cos^2 x}\ ;\ C(x) = \tan x$$

$$y = \sin x + C\cos x.$$

(c) Answer: $y = C\ln^2 x - \ln x.$

2. (a) $u' = u + u^2 e^t$

$$y = u^{-1};\quad u = y^{-1},\quad u' = -\frac{y'}{y^2}$$

$$-\frac{y'}{y^2} = \frac{1}{y} + \frac{1}{y^2}e^t$$

$$-y' = y + e^t$$

homogeneous: $y' = -y$, $y = Ce^{-t}$

inhomogeneous: $y_p = Ae^t$

$-A = A + 1 \Rightarrow A = -\frac{1}{2}$

$$y = -\frac{1}{2}e^t + Ce^{-t}$$

$$u = \frac{1}{-\frac{1}{2}e^t + Ce^{-t}}.$$

(b) $u' = -\frac{1}{t}u + \frac{1}{t u^2}$; $n = -2$

$$y = u^{1-n} = u^3$$

$$y' = 3u^2 u'$$

$$3u^2 u' = -\frac{1}{t} 3u^3 + 3\frac{1}{t}$$

$$y' = -\frac{3}{t}y + \frac{3}{t}$$

homogeneous: $y' = -\frac{3}{t}y$

$$y = Ct^{-3}$$

inhomogeneous: $C'(t)t^{-3} = \frac{3}{t}$

$$c'(t) = 3t^2$$

$$C(t) = t^3 + C_1$$

$$y = 1 + Ct^{-3}$$

$$u = \left(1 + Ct^{-3}\right)^{1/3}.$$

(c) Answer: $y = \left(Ce^{-2x} - e^x\right)^{-1}.$

4. $V(t)$ – volume of the tank.

$$V(t) = 60 - 3t + 2t = 60 - t \quad [gal.]$$

$u(t)$ – mass of salt [lb.].

Balance: $$u(t+\Delta t) = u(t) + \underbrace{2\ gal/min \cdot 1\ lb/gal \cdot \Delta t\ min}_{\text{inflow during } \Delta t}$$

(Δt – small time interval)

$$- \underbrace{3\ gal/min \cdot \frac{u(t)}{V(t)}\ lb/gal \cdot \Delta t\ min}_{\text{outflow during } \Delta t}$$

$$\frac{u(t+\Delta t) - u(t)}{\Delta t} = 2 - 3\frac{u(t)}{V(t)}$$

let $\Delta t \to 0$:

$$u'(t) = 2 - 3\frac{u(t)}{V(t)}$$

$$u'(t) = 2 - \frac{3u(t)}{60-t} \quad \text{(linear)}$$

homogeneous:
$$u'(t) = -\frac{3u(t)}{60-t}$$

$$\ln|u| = 3\ln|60-t| + C$$

$$u = C(60-t)^3.$$

inhomogeneous:

$$C'(t)(60-t)^3 = 2$$

$$C'(t) = \frac{2}{(60-t)^3}$$

$$C(t) = -\frac{1}{(60-t)^2}$$

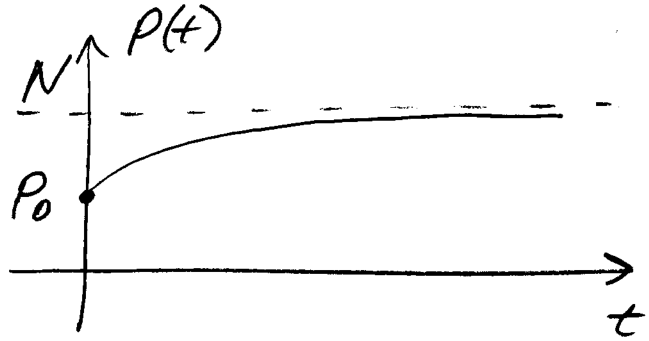
$$u(t) = c(60-t)^3 - (60-t)$$

$$u(0) = 0 \Rightarrow C = \frac{1}{60^2}. \quad u(t) = \frac{1}{360}(60-t)^3 - (60-t),$$
$$0 \le t \le 60.$$

5. $P(t)$ - number of people who heard the rumor.

$$P'(t) = k\,(\underbrace{N - P(t)}_{\text{number of people who have not heard the rumor.}})$$

Solution: $P(t) = N + (P_0 - N)e^{-kt}$.



Over a long time $P(t)$ approaches its limit $= N$.

6. Picard iterates:

$$u_0 = 0;$$

$$u_1(t) = \int_0^t 1\,ds = t$$

$$u_2(t) = \int_0^t (1 + u_1^2(s))\,ds = t + \frac{t^3}{3}$$

$$u_3(t) = \int_0^t (1 + u_2^2(s))\,ds$$

$$= \int_0^t \left(1 + s^2 + \frac{2s^4}{3} + \frac{s^6}{9}\right) ds$$

$$= t + \frac{t^3}{3} + \frac{2}{3 \cdot 5} t^5 + \frac{1}{7 \cdot 9} t^9$$

Exact solution of the DE

$$u(t) = \tan t.$$

8. $u' = -ru, \quad u(0) = u_0.$

$$u(t) = u_0 e^{-rt}$$

Euler method: u_0 — initial data;

$0 \quad h \quad 2h \quad 3h \quad \dots$

$$t_n = nh, \quad j = 0, 1, 2 \dots$$

$$u_1 = (1-hr)u_0$$

$$u_2 = (1-hr)u_1 = (1-hr)^2 u_0$$

$$u_3 = (1-hr)u_2 = (1-hr)^3 u_0$$

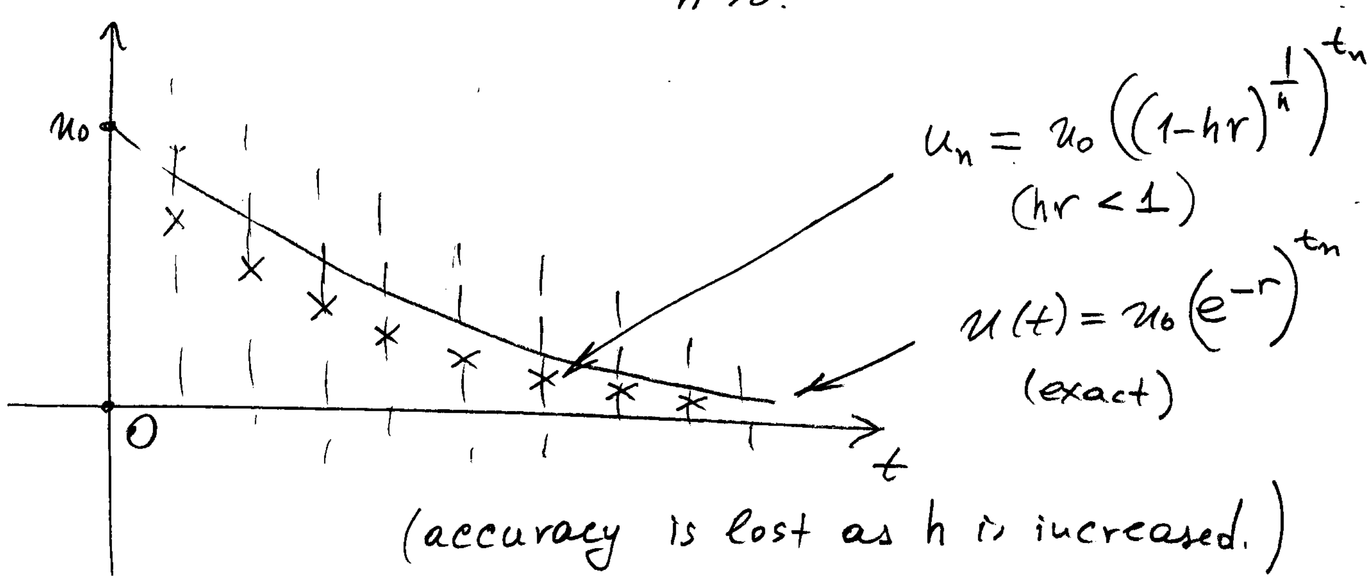
$$\vdots$$

$$u_n = (1-hr)^n u_0 = \left((1-hr)^{\frac{1}{h}}\right)^{t_n} u_0 .$$

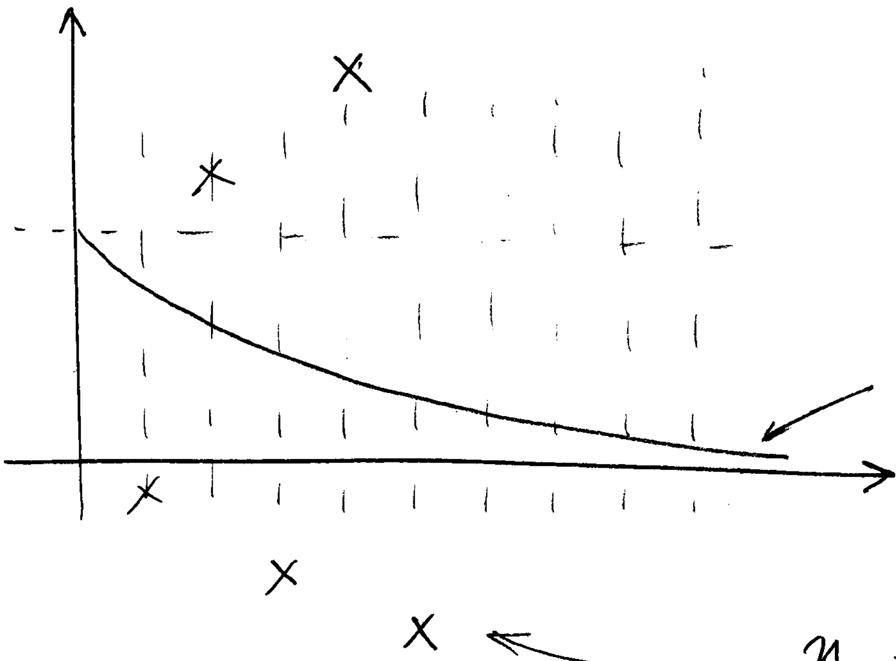
Exact solution $u(t_n) = e^{-rt_n} u_0$

Since $(1-hr)^{\frac{1}{h}} \xrightarrow[h \to 0]{} e^{-r}$, we have

$$u(t_n) - u_n \xrightarrow[h\to 0]{} 0 \qquad (t_n \leq T)$$



(accuracy is lost as h is increased.)



$$u(t) = u_0 \left(e^{-r}\right)^{t_n}$$

$$u_n = u_0 \left((1-hr)^{\frac{1}{h}} \right)^{t_n}$$

$$(hr > 1)$$

Approximate solution becomes crazy (negative values, growing amplitude) if $hr > 1$.

10.

$$3(1-t^2)u' + 2tu = 2t$$

homogeneous:

$$3(1-t^2)u' + 2tu = 0$$

$$\frac{u'}{u} = \frac{-2t}{3(1-t^2)}$$

$$\ln|u| = \frac{1}{3}\ln|1-t^2| + C$$

$$u = C(1-t^2)^{1/3}.$$

inhomogeneous:

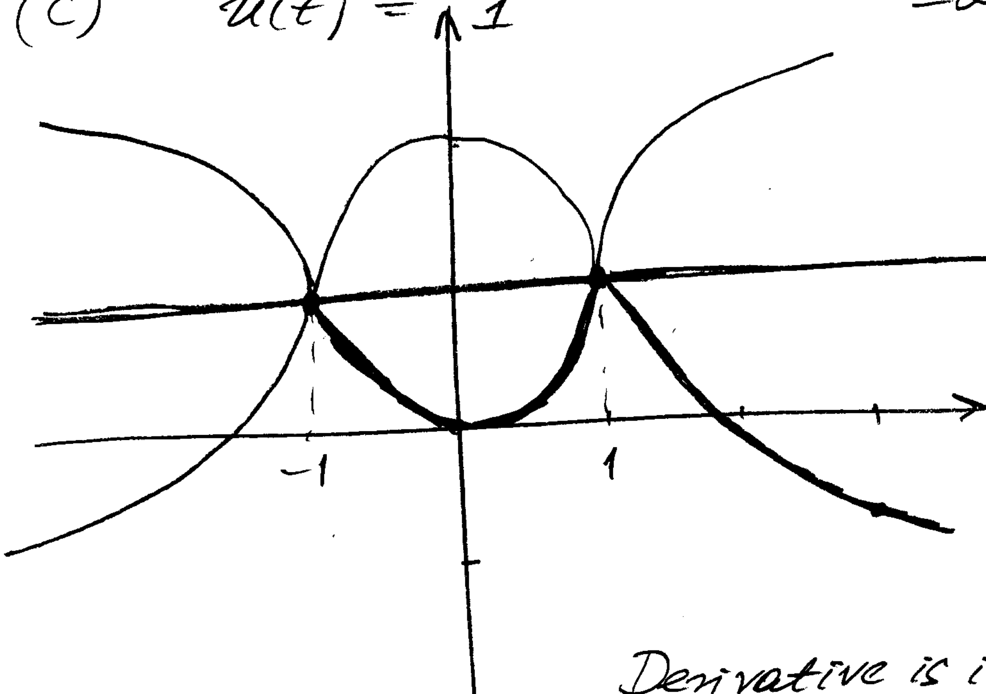
$$u_{part}(t) = 1$$

$$u(t) = 1 + C(1-t^2)^{1/3}.$$

(a) $u(t) = 1 - (1-t^2)^{1/3}$, $\quad -1 < t < 1$

(b) $u(t) = 1 + (1-t^2)^{1/3}$ $\quad t \in (1, \infty)$

(c) $u(t) = 1$ $\quad -\infty < t < \infty$



Derivative is infinite at $t = \pm 1$, unless $C = 0$.