

**Sample final exam**

1. For the system

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$$

- Find the general solution
  - Find equilibrium solutions and determine their stability
  - Sketch the phase portrait
2. Solve the initial-value problem

$$y' = \frac{x^2}{y(1+x^3)}, \quad y(0) = -2.$$

Find the maximal interval on which the solution is defined.

3. Consider the linear system

$$x' = 4x - 2x^2 - xy, \quad y' = y - y^2 - 2xy.$$

Find all equilibrium points and determine the type and stability of the equilibrium point  $(2, 0)$ .

4. Give an example of a
- $2 \times 2$
- linear system for which

- The origin is an unstable node and the lines in the directions  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  are linear orbits.
- The origin is a stable degenerate node and the line in the direction of  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  is the only linear orbit.

5. Consider the initial-value problem

$$u' = u^2 + t, \quad u(1) = 0.$$

Pick step size  $h = 0.5$  and use the forward Euler's method to find an approximation of  $u(2)$ .

6. Find all solutions of the boundary-value problem

$$u'' + u = 1, \quad u(0) = 0, \quad u(2\pi) = 0.$$

7. Consider an  $RC$  circuit composed of a resistor with  $R = 1$  and a capacitor with  $C = 0.25$  connected in series with a  $10V$  battery. Initially the charge on the capacitor is zero. Find the charge  $q(t)$  and the current  $I(t)$  for  $t > 0$ .
8. Find a Cauchy-Euler equation for which  $u(t) = t^{-3} \sin(2t)$  is a solution.
9. Prove that if  $q(t) > 0$  then the ratio  $u'(t)/u(t)$  is a decreasing function of  $t$  for any solution  $u(t)$  of the equation

$$u'' + q(t)u = 0$$

on any interval of  $t$  on which  $u(t) \neq 0$ .

**Answers:**

1. (a)  $\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} \cos t \\ 2 \cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ -\cos t + 2 \sin t \end{pmatrix}$ ; (b)  $(0, 0)$ , stable spiral.
2.  $y(x) = -\sqrt{2 \ln(1 + x^3) + 4}$ ,  $x > -1$ .
3.  $(0, 0)$ ,  $(0, 1)$ ,  $(2, 0)$ ; stable node.
4. (a)  $\mathbf{x}' = \begin{pmatrix} 4 & 0 \\ 1 & 3 \end{pmatrix} \mathbf{x}$ ; (b)  $\mathbf{x}' = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \mathbf{x}$ .
5. 1.375.
6.  $u(x) = 1 - \cos x + C \sin x$ .
7.  $q(t) = \frac{5}{2}(1 - e^{-4t})$ ,  $I(t) = 10e^{-4t}$ .
8.  $t^2 u'' + 7t u' + 13u = 0$ .