Tue, Apr 13, 2010 **MATH 351** **Midterm 2**

Name: (print) ______________________

CSUN ID No. : Solutions.

This exam includes 8 questions (the last one is a bonus). Please check that your copy has 9 pages. The duration of the exam is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	8	total

Important: The exam is closed books/notes. Graphing calculators are permitted. Show all your work.

1. (5 points) Find a second order linear differential equation such that $u_1(t) = \cos(5\ln t)$, $u_2(t) = \sin(5\ln t)$ are two linearly independent solutions.

Cauchy-Euler equation,

char. equation has roots $m_{1,2} = \pm 5i$

$$m^2 + 25 = 0$$

$$m(m-1) + m + 25 = 0$$

$$t^2 u'' + t u' + 25 u = 0.$$

(Check: $u_1'(t) = -\frac{5}{t}\sin(5\ln t)$

$u_1''(t) = \frac{5}{t^2}\sin(5\ln t) - \frac{25}{t^2}\cos(5\ln t)$

$t^2 u_1'' + t u_1' = -25\cos(5\ln t) = -25 u_1$ ✓

likewise for $u_2(t)$.)

2. (10 points)

(a) Formulate the conditions under which all solutions of the equation $a(t)u''+b(t)u'+c(t)u=0$ on the interval (t_1,t_2) are given by

$$u(t)=c_1u_1(t)+c_2u_2(t),$$

where $u_1(t)$, $u_2(t)$ are two arbitrary linearly independent solutions.

$\frac{b(t)}{a(t)}$; $\frac{c(t)}{a(t)}$ continuous on (t_1, t_2).

(b) Consider the functions $u_1(t)=t^3$, $u_2(t)=|t|^3$, and $u_3(t)=t$. Show that they are linearly independent on $(-1,1)$.

Assume $c_1 t^3 + c_2 |t|^3 + c_3 t = 0$, $t \in (-1,1)$.

If $t>0$, $(c_1+c_2)t^3 + c_3 t = 0$

$\Rightarrow c_1 + c_2 = 0,\ c_3 = 0.$

If $t<0$, $(c_1 - c_2)t^3 + c_3 t = 0$

$\Rightarrow c_1 - c_2 = 0,\ c_3 = 0.$

$c_1 + c_2 = 0$
$c_1 - c_2 = 0$ $\Rightarrow c_1 = c_2 = 0.$

Continued...

(c) Find a differential equation of the form $a(t)u'' + b(t)u' + c(t)u = 0$ such that $u_1(t) = t^3$, $u_2(t) = |t|^3$, and $u_3(t) = t$ are solutions for $t \in (-1, 1)$. [Hint: A Cauchy-Euler equation would do.]

t^m, $m = 1, 3$ solutions;

$(m-1)(m-3) = m(m-1) - 3m + 3$

$t^2 u'' - 3tu' + 3u = 0$

$u_2(t) = \begin{cases} t^3, & t > 0 \\ -t^3, & t < 0 \end{cases} \Rightarrow$ solution on $(0,1)$ and on $(-1,0)$.

Since $u_2(t)$ is continuously differentiable at 0, it is a solution on $(-1,1)$.

(d) Which of the conditions of part (a) are not satisfied by the differential equation in part (c)?

$$\left.\begin{aligned} \frac{b(t)}{a(t)} &= -\frac{3}{t} \\ \frac{c(t)}{a(t)} &= \frac{3}{t^2} \end{aligned}\right\} \text{ not continuous at } t = 0.$$

3. (8 points) Solve the initial-value problem

$$u'' + 2u' + u = 2e^{-t}, \quad u(0) = 1, \quad u'(0) = 2.$$

homogeneous problem:

$$u'' + 2u' + u = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda+1)^2 = 0 \Rightarrow \lambda = -1. \quad (\text{multiplicity } 2.)$$

$$u_h(t) = c_1 e^{-t} + c_2 t e^{-t}.$$

particular solution:

$$u_p(t) = a t^2 e^{-t}$$

(t^2 is the lowest power such that $u_p(t)$ is not part of homogeneous solution.)

$$u_p' = -at^2 e^{-t} + 2at e^{-t}$$

$$u_p'' = at^2 e^{-t} - 4at e^{-t} + 2ae^{-t}$$

$$u_p'' + 2u_p' + u_p = 2ae^{-t} = 2e^{-t}$$

$$\Rightarrow a = 1$$

Answer:

$$u(t) = u_p(t) + u_h(t)$$

$$= t^2 e^{-t} + c_1 e^{-t} + c_2 t e^{-t}$$

4. (10 points) Find all values of λ for which the boundary-value problem

$$-u'' = \lambda u, \quad 0 < x < \pi$$
$$u'(0) = 0, \quad u(\pi) = 0$$

has nontrivial solutions, and determine those solutions.

Equation $-u'' = \lambda u$ has solutions

$$u(x) = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x) \quad (\lambda > 0)$$
$$u(x) = C_1 + C_2 x \quad (\lambda = 0)$$
$$u(x) = C_1 e^{\sqrt{-\lambda}\, x} + C_2 e^{-\sqrt{-\lambda}\, x} \quad (\lambda < 0)$$

In the last two cases the only solution that satisfies boundary conditions is zero:

$\lambda = 0$:

$u'(x) = C_2$; $u'(0) = C_2 = 0$

$u(x) = C_1$; $u(\pi) = C_1 = 0$

$\lambda < 0$

$$u'(x) = C_1 \sqrt{-\lambda}\, e^{\sqrt{-\lambda}\, x} - C_2 \sqrt{-\lambda}\, e^{-\sqrt{-\lambda}\, x}$$
$$u'(0) = C_1 \sqrt{-\lambda} - C_2 \sqrt{-\lambda}$$
$$u(\pi) = C_1 e^{\sqrt{-\lambda}\,\pi} + C_2 e^{-\sqrt{-\lambda}\,\pi}$$
$$\begin{vmatrix} \sqrt{-\lambda} & -\sqrt{-\lambda} \\ e^{\sqrt{-\lambda}\,\pi} & e^{-\sqrt{-\lambda}\,\pi} \end{vmatrix} \neq 0$$
$$\Rightarrow C_1 = C_2 = 0$$

If $\lambda > 0$,

$$u(x) = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$$
$$u'(x) = -C_1 \sqrt{\lambda} \sin(\sqrt{\lambda}\, x) + C_2 \sqrt{\lambda} \cos(\sqrt{\lambda}\, x)$$
$$u'(0) = C_2 \sqrt{\lambda} = 0 \Rightarrow C_2 = 0$$
$$u(\pi) = C_1 \cos(\sqrt{\lambda}\, \pi) = 0$$
$$\sqrt{\lambda}\,\pi = \frac{\pi}{2} + \pi n \Rightarrow \lambda = \left(\frac{1}{2} + n\right)^2, \quad n = 0, 1, 2 \ldots$$

$$u(x) = C_1 \cos\left(\left(\frac{1}{2} + n\right) x\right), \quad n = 0, 1, 2 \ldots$$

Continued...

5. (10 points) Find the general solution of $xy'' + 2y' - xy = 0$ $(x > 0)$ given that $y_1(x) = \frac{e^x}{x}$ is a particular solution.

$$y(x) = y_1(x)\, v(x)$$

$$y' = y_1' v + y_1 v'$$
$$y'' = y_1'' v + 2y_1' v' + y_1 v''$$

$$xy'' + 2y' - xy = x(2y_1' v' + y_1 v'') + 2y_1 v' = 0$$

$$v' = w \Rightarrow \quad w' + \left(\frac{2}{x} + 2\frac{y_1'}{y_1}\right) w = 0$$

$$\int \frac{w'\,dx}{w} = -\int \left(\frac{2}{x} + 2\frac{y'}{y}\right) dx$$

$$\ln w = -2 \ln x - 2 \ln y_1$$

$$w = \frac{1}{y_1^2} e^{-2\ln x} = \frac{1}{y_1^2 x^2} = \frac{1}{e^{2x}}$$
$$= e^{-2x}$$

$$v = \int w\, dx = -\frac{1}{2} e^{-2x}$$

$$y_2(x) = -\frac{1}{2} \frac{e^{-x}}{x}$$

$$y(x) = c_1 \frac{e^x}{x} + c_2 \frac{e^{-x}}{x}$$

– general solution.

Continued...

6. (10 points) Use the power series method to find the first three terms of two linearly independent solutions to $u'' + tu' + tu = 0$ valid near $t = 0$.

$$u(t) = \sum_{n=0}^{\infty} a_n t^n \quad ; \quad t u(t) = \sum_{n=1}^{\infty} a_{n-1} t^n$$

$$u'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

$$t u'(t) = \sum_{n=1}^{\infty} n a_n t^n$$

$$u''(t) = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n$$

$$u'' + tu' + tu = 2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + n a_n + a_{n-1} \right] t^n = 0$$

$a_2 = 0,$

$n = 1: \quad 3 \cdot 2 \cdot a_3 + a_1 + a_0 = 0 \Rightarrow a_3 = -\frac{1}{6} a_0 - \frac{1}{6} a_1$

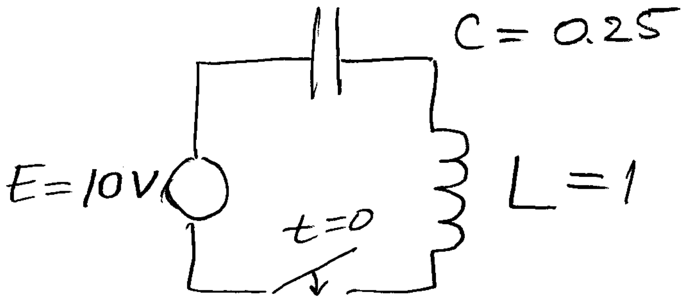
$n = 2: \quad 4 \cdot 3 \cdot a_4 + 2a_2 + a_1 = 0 \Rightarrow a_4 = -\frac{1}{12} a_1$

$n = 3: \quad 5 \cdot 4 \cdot a_5 + 3a_3 + a_2 = 0 \Rightarrow a_5 = \frac{1}{40} a_0 + \frac{1}{40} a_1$

$$u(x) = a_0 \left(1 - \frac{1}{6} x^3 + \frac{1}{40} x^5 + \dots \right) + a_1 \left(x - \frac{1}{6} x^3 - \frac{1}{12} x^4 + \dots \right)$$

Continued...

7. (10 points) An LC circuit contains a capacitor with $C = 0.25$ and an inductor with $L = 1$ connected in series with a $10V$ battery. Find the current $I(t) = q'(t)$ for $t > 0$ if at time $t = 0$ both the current and the charge $q(t)$ are zero. Sketch the graph of the function $I(t)$. [Reminder: the voltage drop on a capacitor is $V_C = \frac{q}{C}$ and the voltage drop on an inductor is $V_L = LI'$.]



$$I(t) = q'(t)$$

$$Lq'' + \frac{1}{C}q = E$$

$$q'' + 4q = 10$$

Particular solution : $q_p = \frac{10}{4} = \frac{5}{2}$

Homogeneous soln: $q_h(t) = C_1\cos(2t) + C_2\sin(2t)$

$$q(t) = q_p + q_h = \frac{5}{2} + C_1\cos(2t) + C_2\sin(2t)$$

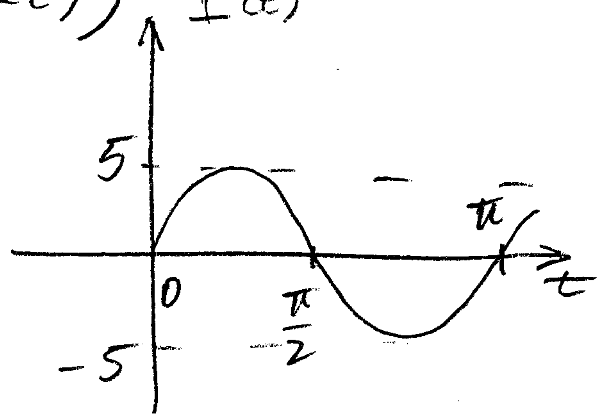
$$q(0) = \frac{5}{2} + C_1 \Rightarrow C_1 = -\frac{5}{2}$$

$$q'(t) = 5\sin(2t) + 2\cdot C_2\cos(2t)$$

$$q'(0) = 2C_2 = 0 \Rightarrow C_2 = 0.$$

$$q(t) = \frac{5}{2}(1 - \cos(2t))$$

$$I(t) = 5\sin(2t)$$



8. (bonus: 10 points) Solve the Cauchy-Euler equation $t^2u'' - 3tu' + 3u = \frac{1}{t^2}$.

$$m(m-1) - 3m + 3 = 0$$
$$m^2 - 4m + 3 = 0$$
$$(m-1)(m-3) = 0$$
$$m_{1,2} = 1, 3$$

homogeneous soln: $u_h = c_1 t + c_2 t^3$.

variation of parameters: $u_p = c_1(t)t + c_2(t)t^3$

$$c_1' t + c_2' t^3 = 0$$
$$c_1' t' + c_2' \cdot (t^3)' = \frac{1}{t^4}$$

$$c_1' = \frac{\begin{vmatrix} 0 & t^3 \\ \frac{1}{t^4} & 3t^2 \end{vmatrix}}{\begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix}} = \frac{-1}{2t^4} \Rightarrow c_1 = \frac{1}{6t^3}$$

$$c_2' = \frac{\begin{vmatrix} t & 0 \\ 1 & \frac{1}{t^4} \end{vmatrix}}{2t^3} = \frac{1}{2t^6} \Rightarrow c_2 = -\frac{1}{10t^4}$$

$$u_p = \frac{1}{6t^3} \cdot t - \frac{1}{10t^4} \cdot t^3 = \left(\frac{1}{6} - \frac{1}{10}\right)\frac{1}{t^2}$$
$$= \frac{1}{15t^2}.$$

Alternatively, undetermined coefficients:

$$u_p = \frac{c}{t^2}; \quad u_p' = -\frac{2c}{t^3}, \quad u_p'' = \frac{6c}{t^4}$$

$$t^2 u_p'' - 3tu_p' + 3u_p = \frac{15c}{t^2} = \frac{1}{t^2} \Rightarrow c = \frac{1}{15}$$

Answer: $u(t) = \frac{1}{15t^2} + c_1 t + c_2 t^3$.

The end.