Tue, Mar 2, 2010 **MATH 351** **Midterm 1**

Name: (print) Solutions.

CSUN ID No. : ______________________

This exam includes 8 questions (the last one is a bonus). Please check that your copy has 8 pages. The duration of the exam is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	8	total

Important: The exam is closed books/notes. Graphing calculators are permitted. Show all your work.

1. (5 points) Verify that $u(t) = 1 + (t^2 + C)^3$ is a one-parameter family of solutions to a DE

$$u' = 6t(u-1)^{2/3}.$$

Show that this equation also has a particular solution that is not obtained from the above family by any choice of C.

$$u(t) = 1 + (t^2+C)^3$$

$$u'(t) = 3(t^2+C)^2 \cdot 2t$$

$$= 6t(t^2+C)^2$$

$$= 6t(u-1)^{2/3} \quad \checkmark$$

$u(t) \equiv 1$ is a solution not obtained from the family.

2. (8 points) Consider the initial-value problem

$$u' = t(u+1), \qquad u(0) = 1.$$

(a) Find the exact solution.

$$\frac{u'}{u+1} = t \; ; \quad \int \frac{du}{u+1} = \int t\,dt = \frac{t^2}{2} + C$$

$$\ln|u+1| = \frac{t^2}{2} + C$$

$$|u+1| = e^{\frac{t^2}{2} + C}$$

$$u+1 = Ce^{\frac{t^2}{2}}$$

$$u = Ce^{\frac{t^2}{2}} - 1$$

$$u(0) = C - 1 = 1 \Rightarrow C = 2$$

$$u(t) = 2e^{\frac{t^2}{2}} - 1$$

(b) Apply Picard iteration with $u_0 = 1$ and compute u_1, u_2, u_3. If the process continues to what function will the approximations converge?

$$u_0 = 1$$

$$u_1 = 1 + \int_0^t 2s\,ds = 1 + t^2$$

$$u_2 = 1 + \int_0^t s(2+s^2)\,ds = 1 + t^2 + \frac{t^4}{4}$$

$$u_3 = 1 + \int_0^t s\left(2 + s^2 + \frac{s^4}{4}\right)ds$$

$$= 1 + t^2 + \frac{t^4}{4} + \frac{t^6}{4\cdot 6}.$$

Expand $$2e^{\frac{t^2}{2}} - 1 = 2\left(1 + \frac{t^2}{2} + \frac{t^4}{2\cdot 4} + \frac{1}{6}\frac{t^6}{8} + \ldots\right) - 1$$

$$= 1 + t^2 + \frac{t^4}{4} + \frac{t^6}{6\cdot 4} + \ldots$$

$$\lim_{n\to\infty} u_n(t) = 2e^{\frac{t^2}{2}} - 1.$$

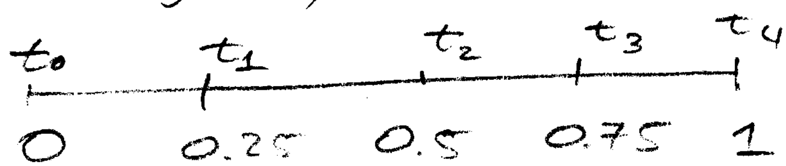
Continued...

3. (5 points) For the intial-value problem in problem 2 use forward Euler's method with $h = 0.25$ to approximate $u(1)$.

$$u' = f(t, u), \quad u(0) = 1$$

$$u_n = u_{n-1} + h\, f(t_{n-1}, u_{n-1}); \quad u_0 = 1$$

$$f(t, u) = t(u+1)$$



$$u_1 = u_0 + h t_0 (u_0 + 1) = 1.$$

$$u_2 = u_1 + h t_1 (u_1 + 1) = 1.125$$

$$u_3 = u_2 + h t_2 (u_2 + 1) = 1.390625$$

$$u_4 = u_3 + h t_3 (u_3 + 1) = 1.83887\ldots$$

(exact value $u(1)$ is $2.29744\ldots$)

4. (8 points) Find the solution of the initial-value problem

$$tu' = u + te^{-u/t}, \quad u(1) = 0$$

and determine the maximal interval on which the solution is defined and continuously differentiable.

$$u' = \frac{u}{t} + e^{-\frac{u}{t}}$$

$$, \quad y = \frac{u}{t}$$

$$u = ty$$

$$u' = y + ty'$$

$$y + ty' = y + e^{-y}$$

$$ty' = e^{-y}$$

$$e^{y} y' = \frac{1}{t}$$

$$e^{y} = \ln|t| + C$$

$$y = \ln(\ln|t| + c)$$

$$u = t \ln(\ln|t| + c)$$

$$u(1) = \ln c = 0 \Rightarrow c = 1$$

$$\ln t + 1 > 0$$

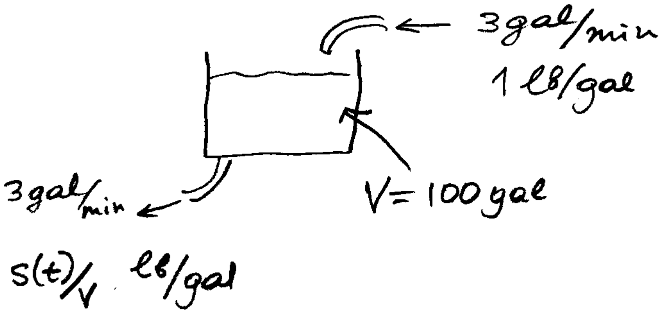
$$\ln t > -1$$

$$t > \frac{1}{e}$$

$$u(t) = t \ln(\ln t + 1), \quad t \in \left(\frac{1}{e}, \infty\right).$$

(solution $u(t)$ goes to $-\infty$ as $t \searrow \frac{1}{e}$.)

5. (8 points) Initially, a tank contains 100 gal of pure water. Then brine containing 1 lb of salt per gallon enters the talk at a rate 3 gal/min. The perfectly mixed solution is drained at the same rate. Set up a differential equation for the amount of salt in the tank and find its solution. Sketch the graph of the solution. When will the amount of salt in the tank reach 0.5 lb?



$s(t)$ – amount of salt in the tank.

$s(0) = 0$ – initial condition;

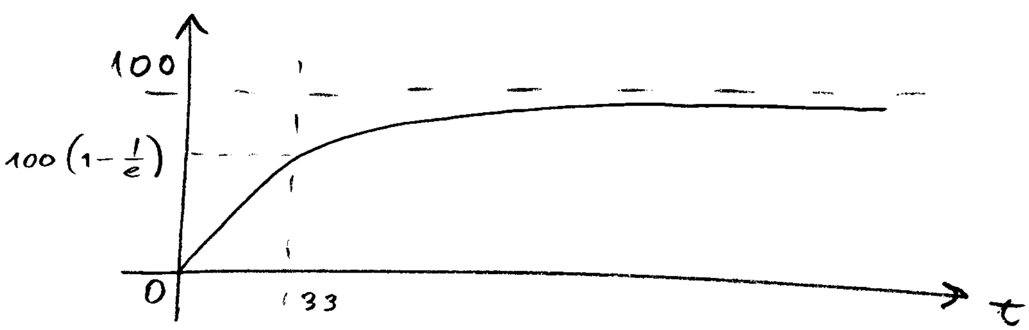
$$s(t+\Delta t) = s(t) + 3\cdot\Delta t - 3\,\frac{s(t)}{V}\,\Delta t$$

$$\overset{\Delta t \to 0}{\Rightarrow} \quad s'(t) = 3 - 3\,\frac{s(t)}{100}$$

$$s(t) = 100 + Ce^{-\frac{3}{100}t}$$

$$s(0) = 0 \Rightarrow C = -100$$

$$s(t) = 100\left(1 - e^{-\frac{3}{100}t}\right)$$



$$100\left(1 - e^{-\frac{3}{100}t}\right) = 0.5$$

$$1 - e^{-\frac{3}{100}t} = 0.005$$

$$e^{-\frac{3}{100}t} = 0.995$$

$$-\frac{3}{100}t = \ln 0.995$$

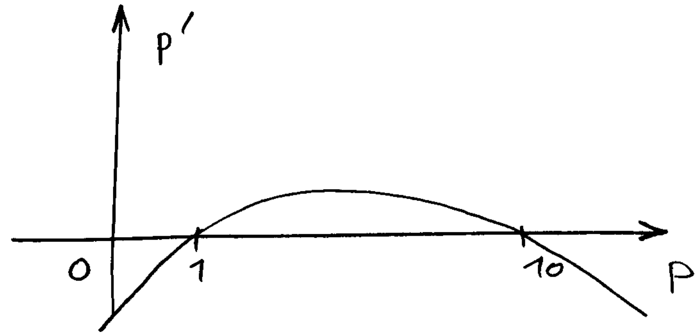
$$t = -\frac{100}{3}\ln 0.995 \approx 0.167\ldots \text{ min} \approx 10 \text{ sec.}$$

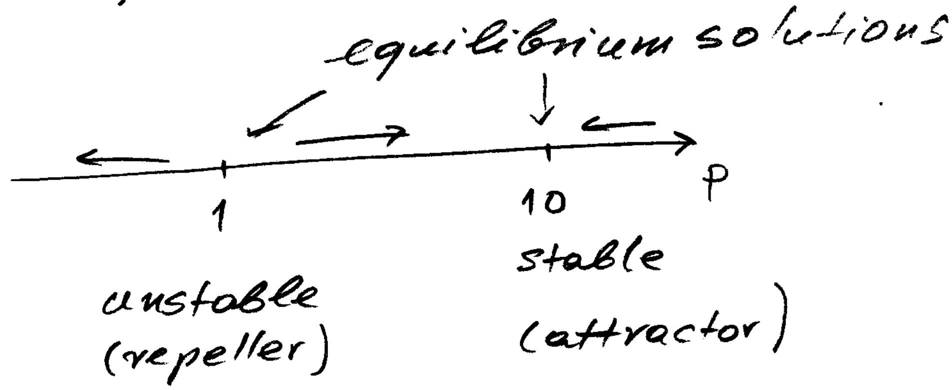
Continued...

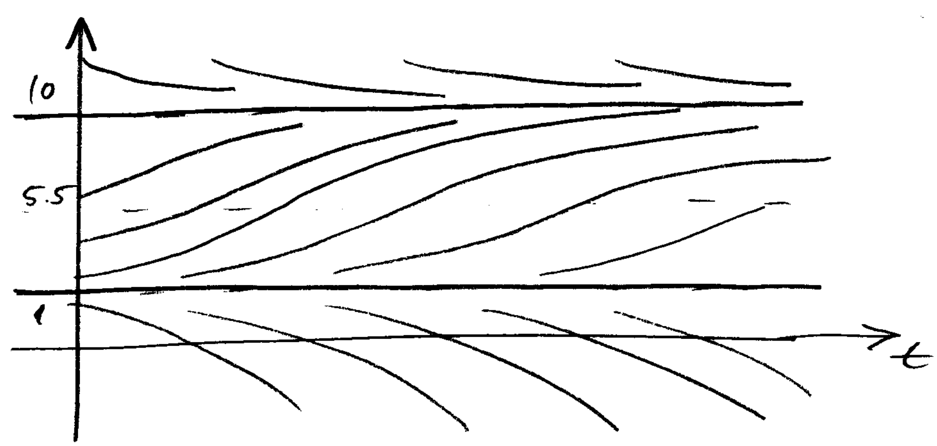
6. (8 points) The differential equation

$$p' = r(p-1)\left(1 - \frac{p}{10}\right)$$

appears as a model for dynamics of populations. Sketch the graph of the right side of the equation as a function of p. Draw a phase line showing equilibrium solutions and determine their stability. Sketch a graph in the (t, p)-plane showing generic solution curves. According to this model, under what conditions does a population become extinct?







If $p(t_0) < 1$ at some moment t_0 then it has to become zero in the future.

7. (10 points) Find general solutions for the following equations

(a) $y' + 2y\tan x = \sin x$ [Hint: there is one particular solution that is easy to guess.]

(b) $xy^2y' = x^2 + y^3$

(a) $y_p = \cos x$ — particular solution.

$$\left(-\sin x + 2\frac{\sin x}{\cos x} \cdot \cos x = \sin x \quad ✓ \right)$$

homogeneous equation:

$$y' + 2y\tan x = 0$$

$$\ln|y| = \int \frac{y'}{y} = -2\int \tan x\, dx = -2\int \frac{\sin x}{\cos x}\, dx$$

$$= 2\int \frac{d\cos x}{\cos x} = 2\ln|\cos x| + C$$

$$\Rightarrow y_h = C\cos^2 x$$

$$y(t) = y_p(t) + y_h(t) = \cos x + C\cos^2 x.$$

(b) $y' = \frac{1}{x}y + x\frac{1}{y^2}$ — Bernoulli equation

$n = -2$;

$$3y^2y' = \frac{1}{x}\cdot 3y^3 + 3x \qquad v = y^{1-n} = y^3$$

$$v' = \frac{3}{x}v + 3x \qquad v' = 3y^2y'$$

$$v' = \frac{3}{x}v \Rightarrow v = Cx^3$$

variation of parameters $\Rightarrow$ $C'(x)\,x^3 = 3x$

$$C'(x) = 3x^{-2}$$

$$C(x) = \int 3x^{-2}dx = -3x^{-1} + c$$

$$v(x) = -3x^2 + Cx^3$$

$$y(x) = \left(-3x^2 + Cx^3\right)^{1/3}$$

Continued...

8. (bonus: 5 points) Solve

$$y' = \cos^2(y - x).$$

$$y - x = v \quad , \quad y' - 1 = v'$$
$$y' = v' + 1$$

$$v' + 1 = \cos^2 v$$

$$v' = \cos^2 v - 1 = -\sin^2 v$$

$$\int \frac{dv}{\sin^2 v} = -\int dt = -t + C$$

$$-\cot v = -t + C$$

$$\cot v = t + C$$

$$v = \cot^{-1}(t + C).$$

The end.