4. (p. 123) $\quad -u'' = \lambda u, \quad 0 < x < 1$

$u(0) = 0, \quad u(1) = 0.$

First, $$\int_0^1 -u'' u \, dx = \underbrace{\left[-u'u\right]_{x=0}^{x=1}}_{\substack{\shortparallel \\ 0 \text{ because of boundary conditions}}} + \int_0^1 (u')^2 dx$$

$$\Rightarrow \int_0^1 (u')^2 dx = \lambda \int_0^1 u^2 dx$$

$$\Rightarrow \lambda = \frac{\int_0^1 (u')^2 dx}{\int_0^1 u^2 dx} \geq 0$$

Thus, if there is a λ for which there is a nontrivial solution, we must have $\lambda \geq 0$.

If $\lambda = 0$, $\quad -u'' = 0 \Rightarrow u = c_1 x + c_2$;

$u(0) = c_2 = 0$

$u(1) = c_1 + c_2 = 0$ $\quad \Rightarrow c_1, c_2 = 0.$

so there is no nontrivial solution with $\lambda = 0$.

If $\lambda > 0$, set $\lambda = m^2$, then

$u(x) = c_1 \cos(mx) + c_2 \sin(mx)$

$u(0) = c_1 = 0$

$u(1) = c_2 \sin(m) \Rightarrow m = \pi n, \; n = 1, 2, \ldots$ in order to have $c_2 \neq 0$.

Solutions $u(x) = C \sin(\pi n x), \; n = 1, 2 \ldots$

if $\lambda = (\pi n)^2, \; n = 1, 2 \ldots$

(C is an arbitrary constant.)

7. (p. 123) If $\omega = 0$, there is a unique solution

$$u(x) = a\,\frac{L-x}{L} + b\,\frac{x}{L}.$$

If $\omega \neq 0$,

$$u(x) = C_1 \cos(\omega x) + C_2 \sin(\omega x)$$

$$u(0) = C_1 = a$$

$$u(L) = C_1 \cos(\omega L) + C_2 \sin(\omega L) = b$$

if $\sin(\omega L) \neq 0$ we can determine

$$C_2 = \frac{b - a\cos(\omega L)}{\sin(\omega L)}$$

Answer: unique solution if

$$\omega L \neq \pi n, \quad n = \pm 1, \pm 2 \ldots$$

$$\left(\text{same as } \omega^2 = \left(\frac{\pi n}{L}\right)^2, \ n = 1, 2 \ldots\right)$$

8. (p. 124) $u'' + 2u' = -\lambda u, \qquad 0 < x < 1$

$$u(0) = u(1) = 0.$$

Char. equation: $m^2 + 2m + \lambda = 0$

$$m^2 + 2m + 1 + (\lambda - 1) = 0$$

$$m = -1 \pm \sqrt{1-\lambda}$$

Case 1: m_1, m_2 - real, distinct.

$$u(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$u(0) = C_1 + C_2 = 0$$

$$u(1) = C_1 e^{m_1} + C_2 e^{m_2} = 0$$

Determinant $\begin{vmatrix} 1 & 1 \\ e^{m_1} & e^{m_2} \end{vmatrix} \neq 0$ if $m_1 \neq m_2$
$\Rightarrow C_1 = C_2 = 0.$

Case 2: $\quad m_1 = m_2 = -1 \qquad (\lambda = 1)$

$$u(x) = c_1 x e^{-x} + c_2 e^{-x}$$

$$u(0) = c_2 = 0$$

$$u(1) = c_1 e^{-1} + c_2 e^{-1} = 0$$

$$\Rightarrow c_1 = 0.$$

Case 3: $\quad m_{1,2} = -1 \pm bi, \quad b = \sqrt{\lambda - 1} \quad (\lambda > 1)$

$$u(x) = c_1 e^{-x} \cos(bx) + c_2 e^{-x} \sin(bx)$$

$$u(0) = c_1 = 0$$

$$u(1) = c_2 e^{-x} \sin(b) = 0$$

if $\sin(b) = 0$, i.e. $b = \pi n$,

$$\lambda - 1 = (\pi n)^2$$

$$\lambda = 1 + (\pi n)^2, \quad n = 1, 2 ..$$

Then $\quad u(x) = C e^{-x} \sin(\pi n x)$

— nonzero solution.

Answer: $\quad \lambda = 1 + (\pi n)^2, \quad n = 1, 2 \ldots$

2 (a)

$$u' = v$$
$$v' = -\frac{1}{2}v - 2u + 3\sin t$$

$$x = \begin{pmatrix} u \\ v \end{pmatrix}$$
$$x' = \begin{pmatrix} 0 & 1 \\ -2 & -\frac{1}{2} \end{pmatrix} x + \begin{pmatrix} 0 \\ 3\sin t \end{pmatrix}$$

(b)

$$u' = v$$
$$v' = -\frac{1}{t}v - \frac{1}{t^2}\left(t^2 - \frac{1}{4}\right)u$$

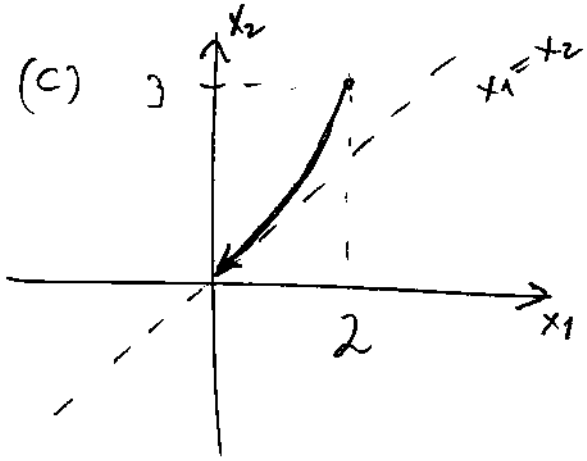
$$x = \begin{pmatrix} u \\ v \end{pmatrix};$$
$$x' = \begin{pmatrix} 0 & 1 \\ -1 + \frac{1}{4t^2} & -\frac{1}{t} \end{pmatrix} x$$

4. Answers:

(a) $x_1(t) = c_1 e^{-t} + c_2 e^{-3t}$;
$x_2(t) = c_1 e^{-t} - c_2 e^{-3t}$

(b) $c_1 = 5/2$; $c_2 = -\frac{1}{2}$

(c)



Approaches the origin tangentially to the line $x_1 = x_2$.

5. Answer:

(a)

$$X_1(t) = c_1 \cos(2t) + c_2 \sin(2t)$$

$$X_2(t) = -c_1 \sin(2t) + c_2 \cos(2t)$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

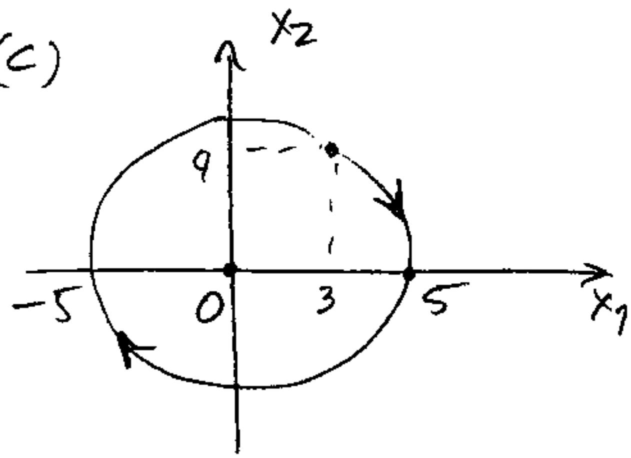
rotation matrix through angle $2t$, clockwise.

(b)

$$x_1(t) = 3\cos(2t) + 4\sin(2t)$$

$$x_2(t) = -3\sin(2t) + 4\cos(2t)$$

(c)



Circle around the origin, traveled clockwise.