1. (p. 115) (a) $t^2u''+3tu'-8u=0$, $u(1)=0$, $u'(1)=2$

Char. equation $m(m-1)+3m-8=0$

$$m^2+2m-8=0$$

$$(m-2)(m+4)=0$$

$$m_1=2,\ m_2=4.$$

$u(t)=c_1t^2+c_2t^4$ - general solution;

$$u(1)=c_1+c_2=0 \Rightarrow c_2=-c_1$$

$$u'(1)=2c_1+4c_2=2 \Rightarrow -2c_1=2$$

$$c_1=-1,$$
$$c_2=1$$

$$u(t)=t^4-t^2.$$

(c) $t^2u''-tu'+2u=0$, $u(1)=0$, $u'(1)=1$

$$m(m-1)-m+2=0$$

$$m^2-2m+2=0$$

$$(m-1)^2+1=0$$

$$m-1=\pm i,\quad m=1\pm i$$

general solution:

$$u(t)=c_1t\cos(\ln t)+c_2t\sin(\ln t).$$

$$u(1)=c_1=0;$$

$$u'(1)=c_1\cos(\ln t)-c_1\sin(\ln t)+c_2\sin(\ln t)+c_2\cos(\ln t)\Big|_{t=1}$$

$$=c_1+c_2=1 \Rightarrow c_2=1$$

$$u(t)=t\sin(\ln t).$$

5. (p. 115) $u'' - tu = 0, \quad t_0 = 0.$

$$u = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 + \dots$$

$$u'' = 2a_2 + 6a_3 t + 12 a_4 t^2 + 20 a_5 t^3 + 30 a_6 t^4 + 42 a_7 t^5 + \dots$$

$$u'' - tu = 2a_2 + (6a_3 - a_0)t + (12a_4 - a_1)t^2 + (20a_5 - a_2)t^3 + (30a_6 - a_3)t^4 + (42a_7 - a_4)t^5 + \dots$$

$$2a_2 = 0 \Rightarrow a_2 = 0$$

$$6a_3 - a_0 = 0 \Rightarrow a_3 = \frac{1}{6} a_0$$

$$12a_4 - a_1 = 0 \Rightarrow a_4 = \frac{1}{12} a_1$$

$$20a_5 - a_2 = 0 \Rightarrow a_5 = 0$$

$$30a_6 - a_3 = 0 \Rightarrow a_6 = \frac{1}{180} a_0$$

$$42a_7 - a_4 = 0 \Rightarrow a_7 = \frac{1}{504} a_1$$

$$u(t) = a_0\left(1 + \frac{1}{6}t^3 + \frac{1}{180}t^6 + \dots\right) + a_1\left(t + \frac{1}{12}t^4 + \frac{1}{504}t^7 + \dots\right)$$

11. (p. 116) $t^2 u'' + t u' + (t^2 - \frac{1}{4}) u = 0$

$$u_1(t) = \frac{1}{\sqrt{t}} \cos t$$

Look for $u_2(t) = v(t)\, u_1(t)$.

Then $u_2' = v' u_1 + v u_1'$

$$u_2'' = v'' u_1 + 2 v' u_1' + v u_1''$$

Then $t^2 (v'' u_1 + 2 v' u_1') + t\, v' u_1 = 0$

$$v'' + \left(\frac{2 u_1'}{u_1} + \frac{1}{t} \right) v' = 0$$

$$v' = C e^{- \int \frac{2 u_1' dt}{u_1} - \int \frac{1}{t} dt}$$

$$= \frac{C}{u_1^2} e^{- \ln t} = \frac{C}{t u_1^2}$$

$$= \frac{C}{\cos^2 t} \leftarrow \left(\begin{array}{l} \text{it is enough} \\ \text{to consider} \\ C = 1 \end{array} \right)$$

$$v(t) = \int \frac{1}{\cos^2 t} dt = \frac{\sin t}{\cos t} + C'$$

$$u_2(t) = \frac{1}{\sqrt{t}} \sin t.$$

15. (p. 116) $u'' + t u' + u = 0$, $u_1(t) = e^{-t^2/2}$

$$v' = \frac{e^{-\int t dt}}{e^{-t^2}} = \frac{e^{-\frac{t^2}{2}}}{e^{-t^2}} = e^{\frac{t^2}{2}}.$$

$$v(t) = \int e^{\frac{t^2}{2}} dt$$

$$u_2(t) = e^{-t^2/2} \int e^{t^2/2} dt$$

$$u(t) = c_1 e^{-t^2/2} + c_2 e^{-t^2/2} \int e^{t^2/2} dt.$$

17 (a) (p. 117)

Notice that there is a typo: DE should read

$$u'' + \frac{1}{t} u' = a$$

$$u(t) = c_1(t) + c_2(t) \ln t$$

Variation of parameters:

$$\begin{cases} c_1'(t) + c_2'(t) \ln t = 0 \\ c_1'(t) \cdot 0 + c_2'(t) \frac{1}{t} = a \end{cases}$$

$$c_2'(t) = at \Rightarrow c_2(t) = \frac{at^2}{2}$$

$$c_1'(t) = -at \ln t$$

$$c_1(t) = -a \int t \ln t \, dt = -a \int \ln t \, d\frac{t^2}{2}$$

$$= -\frac{at^2}{2} \ln t + \frac{a}{2} \int t \, dt$$

$$= -\frac{at^2}{2} \ln t + \frac{at^2}{2} + C$$

$$u_p(t) = \frac{at^2}{2} \ln t - \frac{at^2}{2} \ln t + \frac{at^2}{2}$$

$$= \frac{at^2}{2}.$$

(d) $u'' - u = \frac{1}{t}$; $u_1(t) = e^t$, $u_2(t) = e^{-t}$

$$u_p(t) = c_1(t) e^t + c_2(t) e^{-t}$$

$$c_1' e^t + c_2' e^{-t} = 0$$

$$c_1' e^t - c_2' e^{-t} = \frac{1}{t}$$

$$c_1' = \frac{\begin{vmatrix} 0 & e^{-t} \\ \frac{1}{t} & -e^{-t} \end{vmatrix}}{\begin{vmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{vmatrix}} = \frac{-\frac{1}{t}e^{-t}}{-2}$$

$$= \frac{1}{2t}e^{-t}$$

$$c_2' = \frac{\begin{vmatrix} e^{t} & 0 \\ e^{t} & \frac{1}{t} \end{vmatrix}}{-2} = -\frac{1}{2t}e^{t}$$

$$u_p(t) = e^{t}\int \frac{1}{2t}e^{-t}\,dt - e^{-t}\int \frac{1}{2t}e^{t}\,dt$$

(integrals are not evaluated within elementary functions.)

1. (e) (p. 127)

$$u''' + u'' = 2e^{t} + 3t^{2}$$

homogeneous: $u''' + u'' = 0$

$$\lambda^3 + \lambda^2 = 0$$

$$\lambda^2(\lambda+1) = 0$$

$\lambda_{1,2} = 0$

$\lambda_3 = -1$

$$u_h(t) = c_1 + c_2 t + c_3 e^{-t}.$$

$$u_{p_1}(t) = a e^{t}$$

$$2ae^{t} = 2e^{t} \Rightarrow a = 1.$$

$$u_{p_2}(t) = t^2(at^2 + bt + c)$$

($\lambda = 0$ is eigenvalue of mult. 2.)

$$4 \cdot 3 \cdot 2 \cdot at + 3 \cdot 2\, b + 4 \cdot 3 \cdot at^2 + 3 \cdot 2bt + 2c = 3t^2$$

$$a = \frac{1}{4}, \quad b = -1, \quad c = 3$$

$$u(t) = e^{t} + t^2\left(\frac{1}{4}t^2 - t + 3\right) + c_1 + c_2 t + c_3 e^{-t}.$$

9 (p. 130) $\quad u'' + 4tu' + 2(2t^2+1)u = 0$

$$u_1(t) = e^{-t^2}$$

$$u_2(t) = v(t)e^{-t^2}$$

$$v'(t) = \frac{1}{u_1^2(t)} e^{-\int 4t\,dt} = \frac{e^{-2t^2}}{e^{-2t^2}} = 1$$

$$v(t) = t$$

$$u_2(t) = te^{-t^2}$$

$$u(t) = c_1 e^{-t^2} + c_2 t e^{-t^2}$$

$$u(0) = c_1 = 3$$

$$u'(0) = c_2 e^{-t^2}\Big|_{t=0} = c_2 = 1$$

$$u(t) = 3e^{-t^2} + te^{-t^2}.$$

12. (p. 131) $\quad t^3u''' - t^2u'' + 2tu' - 2u = 0$

$$m(m-1)(m-2) - m(m-1) + 2m - 2 = 0$$

$$(m-1)\big(m(m-2) - m + 2\big) = 0$$

$$(m-1)(m-2)(m-1) = 0 \ ; \quad m_{1,2} = 1,\ m_3 = 2$$

$$u_1(t) = t,\ u_2(t) = t\ln t,\ u_3(t) = t^2.$$

$$u(t) = c_1 t + c_2 t\ln t + c_3 t^2$$

$$u(1) = c_1 + c_3 = 3; \qquad \Rightarrow c_1 = 1$$

$$u'(1) = c_1 + c_2 + 2c_3 = 2 \qquad c_3 = 2$$

$$u''(1) = c_2 + 2c_3 = 1 \qquad c_2 = -3$$

$$u(t) = t(1 - 3\ln t) + 2t^2.$$

6. $a(t)u'' + b(t)u' + c(t)u = f(t)$.

Try to find $a(t), b(t), c(t), f(t)$.

Hope that $c(t) \neq 0$, then divide by $c(t)$;

redefine $a(t) := \frac{a(t)}{c(t)}$, $b(t) := \frac{b(t)}{c(t)}$,

$f(t) := \frac{f(t)}{c(t)}$;

$$a(t)u'' + b(t)u' + u = f(t)$$

Substitute $u(t) = 1$:

$$1 = f(t).$$

Substitute $u(t) = t$

$$b(t) + t = 1 \Rightarrow b(t) = 1 - t.$$

Substitute $u(t) = t^2$:

$$2a(t) + 2t(1-t) + t^2 = 1$$

$$a(t) = \frac{1 - t^2 - 2t(1-t)}{2} = \frac{(1-t)^2}{2}$$

Obtain

$$\frac{(t-1)^2}{2}u'' - (t-1)u' + u = 1.$$

Set $\tau = t - 1$, $v(\tau) = u(t) \Rightarrow$

$$u(t) = v(t-1)$$
$$u'(t) = v'(\tau)$$
$$u''(t) = v''(\tau)$$

$$\frac{\tau^2}{2}v'' - \tau v + v = 1$$

$$\frac{1}{2}m(m-1) - m + 1 = 0$$

$$\frac{1}{2}m^2 - \frac{3}{2}m + 1 = 0$$

$$\frac{1}{2}(m-1)(m-2) = 0, \quad m_1 = 1, \; m_2 = 2$$

$$v(\tau) = c_1 \tau + c_2 \tau^2$$

$$u_h(t) = c_1 (t-1) + c_2 (t-1)^2$$

$$u(t) = 1 + c_1 (t-1) + c_2 (t-1)^2$$

— general solution