8. (p. 95) $u(t) = e^{-3t} + 2te^{-3t}$

Char. equation $(\lambda+3)^2 = 0$

$$\lambda^2 + 6\lambda + 9 = 0$$

$$u'' + 6u' + 9u = 0$$

One possible set of initial conditions is

$$u(0) = 1$$

$$u'(0) = -1$$

(more generally, $u(t_0) = \left(e^{-3t} + 2te^{-3t}\right)\Big|_{t=t_0}$

$u'(t_0) = \left(e^{-3t} + 2te^{-3t}\right)'\Big|_{t=t_0}$.)

12. (p. 95) $5I'' + \frac{1}{2}I = 0$

$$I(0) = 0,\ I'(0) = 1$$

$$I(t) = c_1 \cos\left(\frac{1}{\sqrt{10}}t\right) + c_2 \sin\left(\frac{1}{\sqrt{10}}t\right)$$

$$I(0) = c_1 = 0$$

$$I'(0) = c_2\frac{1}{\sqrt{10}} = 1 \Rightarrow c_2 = \sqrt{10}$$

$$I(t) = \sqrt{10}\sin\left(\frac{1}{\sqrt{10}}t\right).$$

2. If
$$u_1'' + pu_1' + qu_1 = f_1(t)$$
$$u_2'' + pu_2' + qu_2 = f_2(t)$$
Then $u_p(t) = u_1(t) + u_2(t)$ satisfies
$$u_1'' + u_2'' + pu_1' + pu_2' + qu_1 + qu_2 = f_1(t) + f_2(t)$$
$$(u_1+u_2)'' + p(u_1+u_2)' + q(u_1+u_2) = f_1(t) + f_2(t)$$
$$u_p'' + pu_p' + qu_p = f_1(t) + f_2(t).$$

3. (a) Answer: $u(t) = -6t + \frac{1}{2}e^{2t} + c_1 + c_2 e^t$

(b) $u'' - 3u' + 4u = 2t^2 + te^t + 3\sin t$

homogeneous: $\lambda^2 - 3\lambda + 4 = 0$; $(\lambda - 4)(\lambda + 1) = 0$
$$\lambda_1 = 4, \ \lambda_2 = -1.$$
$$u_h(t) = C_1 e^{4t} + C_2 e^{-t}.$$

Find $u_{p_1}(t)$ for $f_1(t) = 2t^2$
$u_{p_2}(t)$ for $f_2(t) = te^t$, $u_{p_3}(t)$ for $f_3(t) = 3\sin t$.

$$u_{p_1}(t) = at^2 + bt + c$$
$$u_{p_1}'' - 3u_{p_1}' + 4u_{p_1} = 2a - 3(2at + b) + 4(at^2 + bt + c) = 2t^2$$
$$a = \frac{1}{2} \ ; \ b = \frac{3}{4} \ ; \ c = \frac{1}{4}(3b - 2a) = \frac{5}{16}$$
$$u_{p_1}(t) = \frac{1}{2}t^2 + \frac{3}{4}t + \frac{5}{16}$$

$$u_{p_2}(t) = (at + b)e^t$$
$$u_{p_2}'' - 3u_{p_2}' + 4u_{p_2} = (2a + at + b)e^t - 3(a + at + b)e^t + 4(at + b)e^t = te^t$$
$$a = \frac{1}{2} \ ; \ b = \frac{1}{4},$$

$$u_{p_2}(t) = \left(\frac{1}{2}t + \frac{1}{4}\right)e^t.$$

$$u_{p_3}(t) = a\cos t + b\sin t$$

$$u_{p_3}'' - 3u_{p_3}' + 4u_{p_3} = 3a\cos t + 3b\sin t + 3a\sin t - 3b\cos t = 3\sin t$$

$$a = b = \frac{1}{2}$$

$$u_{p_3}(t) = \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

$$u(t) = u_{p_1}(t) + u_{p_2}(t) + u_{p_3}(t) + u_h(t).$$

(c) Answer: $u(t) = -\frac{1}{8}\cos(2t) + C_1 e^{2t} + C_2 e^{-2t}$

(d) Answer:

$$u(t) = \frac{1}{8}t\sin(2t) + C_1\cos(2t) + C_2\sin(2t).$$

(e) Hint: Use $\sin^2 t = \dfrac{1-\cos(2t)}{2}$

Solve

$$u'' + u' + 2u = \frac{1}{2} - \frac{1}{2}\cos(2t)$$

as in parts (a), (b).

4. (a) $u'' + u = \tan t$

Homogeneous eqn $u'' + u = 0$

has solutions $u_1 = C_1 \cos t + C_2 \sin t$

Look for $u_p(t) = C_1(t) \cos t + C_2(t) \sin t$

$$u_p'(t) = \overbrace{C_1' \cos t + C_2' \sin t}^{=0}$$

$$+ C_1 (\cos t)' + C_2 (\sin t)'$$

$$u_p''(t) = C_1' (\cos t)' + C_2' (\sin t)'$$

$$+ C_1 (\cos t)'' + C_2 (\sin t)''$$

$$u_p'' + u_p = C_1' (\cos t)' + C_2' (\sin t)' = \tan t$$

$$C_1' \cos t + C_2' \sin t = 0$$

$$C_1' = \frac{\begin{vmatrix} 0 & \sin t \\ \tan t & \cos t \end{vmatrix}}{\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}} = -\frac{\sin^2 t}{\cos t}$$

$$= \frac{\cos^2 t - 1}{\cos t} = \cos t - \frac{1}{\cos t}$$

$$C_2' = \frac{\begin{vmatrix} \cos t & 0 \\ -\sin t & \tan t \end{vmatrix}}{1} = \sin t$$

$$C_1 = \sin t - \ln|\sec t + \tan t| + \tilde{C}_1$$

$$C_2 = -\cos t + \tilde{C}_2$$

$$u_p(t) = -\cos t \ln|\sec t + \tan t|$$

$$u = u_p(t) + C_1 \cos t + C_2 \sin t.$$

4 (в) $u'' - 2u' + u = \frac{e^t}{1+t^2}$

Homogeneous eqn: $u'' - 2u' + u = 0$

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda = 1$ (multiplicity 2)

$u_h(t) = C_1 e^t + C_2 t e^t$.

Look for $u_p(t) = C_1(t) e^t + C_2(t) t e^t$

$u'' - 2u' + u = C_1' e^t + C_2' (te^t)' = \frac{e^t}{1+t^2}$

$C_1' e^t + C_2' t e^t = 0$

$$C_1' = \frac{\begin{vmatrix} 0 & te^t \\ \frac{e^t}{1+t^2} & (t+1)e^t \end{vmatrix}}{\begin{vmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{vmatrix}} = -\frac{t}{1+t^2}$$

$$C_2' = \frac{\begin{vmatrix} e^t & 0 \\ e^t & \frac{e^t}{1+t^2} \end{vmatrix}}{e^{2t}} = \frac{1}{1+t^2}$$

$$C_1 = -\int \frac{t\,dt}{1+t^2} = -\frac{1}{2}\ln(1+t^2) + C$$

$$C_2 = \arctan t + C$$

$u_p(t) = -\frac{1}{2}\left(\ln(1+t^2)\right)e^t + te^t \arctan t$

$u(t) = u_p(t) + C_1 e^t + C_2 t e^t$.

6. $$\Delta(t) = \begin{vmatrix} e^t & \sin t \\ e^t & \cos t \end{vmatrix} = e^t(\cos t - \sin t)$$

$= 0$

whenever $\cos t = \sin t$,

i.e $t = \frac{\pi}{4} + \pi n$, $n = 0, \pm 1, \pm 2 \ldots$

$\Delta(t)$ has the property that $\Delta(t)$ is either $\equiv 0$, or never zero, if the functions $u_1(t)$, $u_2(t)$ - solutions of the same linear homogeneous equation.

This is violated for e^t, $\sin t$ near $t = \frac{\pi}{4}$, etc.

Indeed, it is clear that there is no equation with constant coefficients so that $e^t, \sin t$ are solutions.

Also, if $u_1(t) = e^t$ is a particular solution of

$$u'' + p(t)u' + q(t)u = 0$$

then $u_2(t) = \sin t = v(t)e^t$, where

$$\left(e^{-t}\sin t\right)' = v'(t) = \frac{C}{e^{2t}} e^{-\int p(t)dt}$$

$$e^t(\cos t - \sin t) = Ce^{-\int p(t)dt}$$

$$-\int p(t)dt = t + \underbrace{\ln|\cos t - \sin t|}_{\text{not continuous at } t = \frac{\pi}{4} + \pi n} + C$$

$\Rightarrow p(t)$ is not continuous at $t = \frac{\pi}{4} + \pi n$

$\Rightarrow u_1(t), u_2(t)$ could be solutions of an equation with variable coefficients, but only on one of the intervals

$\left(-\frac{3\pi}{4}, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{5\pi}{4}\right), \left(\frac{5\pi}{4}, \frac{9\pi}{4}\right)$, etc.