Homework Assignment 2

Due Thu. Feb. 11, 2010, in class.

1. (Population dynamics) At low population densities it may be difficult for an animal to reproduce because of a limited number of suitable mates. A population model that predicts this behavior is the Allee model (W. C. Allee, 1885–1955)

$$p' = rp\Big(\frac{p}{a} - 1\Big)\Big(1 - \frac{p}{K}\Big)$$

   Find the per capita growth rate and plot the per capita rate vs. p. Graph p' vs. p, determine the equilibrium populations, and draw the phase line. Which equilibria are attractors and which are repellers? Which are asymptotically stable? From the phase line plot, describe the long time behavior of the system for different initial populations, and sketch generic solution curves for different initial conditions.

2. Consider the following model for the growth of a population of animals $p(t)$:

$$p'(t) = rp(t)\left(1 - \frac{p(t)}{K}\right) - h, \quad p(0) = p_0$$

   The constant $h > 0$ can be interpreted as the *harvesting rate*: the population grows logistically, and at the same time animals are being removed (by hunting, fishing, or whatever) at a constant rate of h animals per unit time. Draw the phase line for the equations, and determine the equilibrium (time-independent) solutions. Which are asymptotically stable? Explain how the system will behave for different initial conditions. Does the population ever become extinct?

3. Find the general solution in explicit form of the following equations.

   (a) $u' = \frac{2u}{t+1}$

   (b) $u' = \frac{t\sqrt{t^2+1}}{\cos u}$

   (c) $u' = (t+1)(u^2+1)$

   (d) $u' + u + \frac{1}{u} = 0.$

4. Find the solution of the initial-value problem

$$u' = t^2 e^{-u}, \quad u(0) = \ln 2,$$

   and determine the maximal interval of existence.

5. Draw the phase line associated with the DE $u' = u(4 - u^2)$ and then solve the DE subject to the initial condition $u(0) = 1$. (Hint: for the integration you will need a partial fractions expansion

$$\frac{1}{u(4-u^2)} = \frac{a}{u} + \frac{b}{2+u} + \frac{c}{2-u},$$

where a, b, and c are to be determined.)

6. In very cold weather the thickness of ice on a pond increases at a rate inversely proportional to its thickness. If the ice initially is 0.05 inches thick and 4 hours later it is 0.075 inches thick, how thick will it be in 10 hours?

7. Write the solution to the initial value problem

$$u' = -u^2 e^{-t^2}, \quad u(0) = \frac{1}{2}$$

in terms of the erf function, $\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2}\, ds$.

8. (Newton's law of cooling) Suppose the temperature inside your winter home is 68°F at 1:00 P.M. and your furnace then fails. If the outside temperature is 10°F and you notice that by 10:00 P.M. the inside temperature is 57°F, what will be the temperature in your home the next morning at 6:00 A.M.?

9. Find the general solution of the linear equations

   (a) $u' = -\frac{1}{t}u + t$

   (b) $u' = -u + e^t$.