Homework Assignment 1

Due Tue. Feb. 2, 2010, in class.

1. Verify that the initial-value problems

$$u' = 1 - u^2, \quad u(0) = 0, \quad \text{and} \quad u' = 1 + u^2, \quad u(0) = 0$$

have solutions $u(t) = \frac{e^{2t}-1}{e^{2t}+1}$ and $u(t) = \tan t$, respectively. What are the maximal intervals on which each of these solutions are defined and continuously differentiable?

2. Show that $u(t) = \ln(t + C)$ is a one-parameter family of solutions of the DE $u' = e^{-u}$ where C is an arbitrary constant. Plot several members of this family. Find and plot a particular solution that satisfies the initial condition $u(0) = 0$.

3. Find a solution $u = u(t)$ of $u' + 2u = t^2 + 4t + 7$ in the form of a quadratic function of t.

4. Plot the one-parameter family of curves $u(t) = (t^2 + C)\, e^t$, and find a differential equation whose solution is this family. [Hint: the solution curves look different for $C > 1$, $C = 1$ and $C < 1$. Show all three types of solutions on your graph.]

5. Classify the first-order equations as linear or nonlinear, autonomous or nonautonomous.
   (a) $u' = 2t^3 u - 6$.
   (b) $(\cos t)u' - 2u \sin u = 0$.
   (c) $u' = 1 - u^2$.
   (d) $7u' - 3u = 0$.

6. Consider the linear differential equation $u' = p(t)u + q(t)$. Is it true that the sum of two solutions is again a solution? Is a constant times a solution again a solution? Answer these same questions if $q(t) = 0$. Show that if u_1 is a solution to $u' = p(t)u$ and u_2 is a solution to $u' = p(t)u + q(t)$, then $u_1 + u_2$ is a solution to $u' = p(t)u + q(t)$.

7. By hand, sketch the direction field for the DE $u' = u(2-u)$ in the window $-2 \le t \le 2$, $0 \le u \le 3$ at half-integer points ($(0, 0)$, $(0, \pm\frac{1}{2})$, $(\pm\frac{1}{2}, 0)$, etc.). What is the value of the slope along the lines $u = 0$ and $u = 2$? Show that $u(t) = 0$ and $u(t) = 2$ are constant solutions to the DE. On your slope field plot, draw in several solution curves.

8. For the DE in the previous problem show that all solutions $u(t)$ are convex ($u'' > 0$) if $u > 2$ or $0 < u < 1$, and they are concave ($u'' < 0$) if $1 < u < 2$. Check your graph from the previous problem to make sure the solution curves satisfy these conditions. Hint: differentiate the DE to find the equation statisfied by the second derivative.

9. For the DE $u' = u + t$ sketch the lines in the (t, u)-plane where the slope of the solution has a constant value (try values -3, -2, -1, 0, 1, 2...). Using the obtained information draw several approximate solution curves. (Lines and curves in the (t, u)-plane where the slope of the solution is constant are called **isoclines**.)

10. Using antiderivatives, find the general solution to the pure time equation $u = t\cos(t^2)$, and then find the particular solution satisfying the initial condition $u(0) = 1$. Graph the particular solution on the interval $[5, 5]$.

11. Solve the initial value problem $u' = \frac{t+1}{\sqrt{t}}$, $u(1) = 4$.

12. Consider a damped spring-mass system whose position $x(t)$ is governed by the equation $mx'' = -cx' - kx$ (c and k are positive constants. Show that this equation can have a "decaying-oscillation" solution of the form $x(t) = e^{-\lambda t}\cos\omega t$. (Hint: By substituting into the differential equations, show that the decay constant λ and frequency ω can be determined in terms of the given parameters m, c, and k.)