Diffusion Equations in Cylindrical Coordinates

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Seminar in Engineering Analysis

February 4, 2009

California State University Northridge

Outline

- Review last class
  - Gradient and convection boundary condition
- Diffusion equation in radial coordinates
- Solution by separation of variables
- Result is form of Bessel's equation
- Review Bessel functions
- Eigenfunction expansion in Bessel functions

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Review Homework Problem

- Page 561 problem 5: find u(x,t) for $0 \le x \le L$

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \qquad \left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0 \qquad u(x,0) = f(x)$$

- Start with separation of variables solution

$$u(x,t) = e^{-\lambda^2 \alpha t}\left[C_1 \sin(\lambda x) + C_2 \cos(\lambda x)\right]$$

$$\partial u/\partial x = \lambda e^{-\lambda^2 \alpha t}\left[C_1 \cos(\lambda x) - C_2 \sin(\lambda x)\right]$$

- For $\partial u/\partial x|_{x=0} = 0$, $C_1 = 0$
- For $\partial u/\partial x|_{x=L} = 0$, $\lambda = n\pi/L$ for integer n

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Review Homework Problem II

- General solution is sum of all eigenfunctions

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-\lambda_n^2 \alpha t} \cos(\lambda_n x) \qquad \lambda_n = \frac{n\pi}{L}$$

0 for cosines

- General orthogonal relationship for A_n

$$A_n = \frac{\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx}{\int_0^L \cos^2\left(\frac{n\pi x}{L}\right) dx}$$

- Final cosine result has A_0

$$A_0 = \frac{1}{L}\int_0^L f(x)dx \qquad A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

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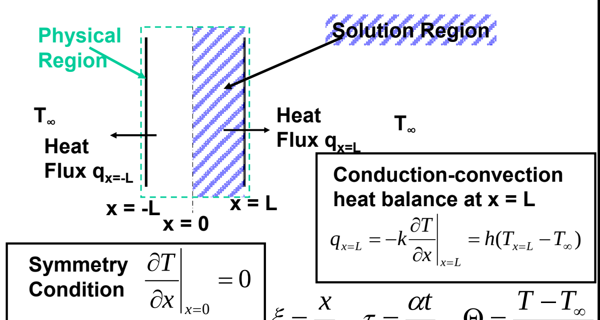
Review Convection Problem



$$\xi = \frac{x}{L} \qquad \tau = \frac{\alpha t}{L^2} \qquad \Theta = \frac{T - T_\infty}{T_0 - T_\infty}$$

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Review Dimensionless Problem

- Diffusion equation $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$ | $\frac{\partial \Theta}{\partial \tau} = \frac{\partial^2 \Theta}{\partial \xi^2}$
- Initial condition $T(x,0) = f_0(x)$ | $\Theta(\xi,0) = \frac{f_0(\xi L) - T_\infty}{T_0 - T_\infty}$
- Symmetry boundary condition $\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$ | $\left.\frac{\partial \Theta}{\partial \xi}\right|_{\xi=0} = 0$
- Convection boundary condition

$$k\left.\frac{\partial T}{\partial x}\right|_{x=L} + h(T_{x=L} - T_\infty) = 0 \qquad \left.\frac{\partial \Theta}{\partial \xi}\right|_{\xi=1} + \frac{hL}{k}\Theta_{\xi=1} = 0$$

Homogenous boundary condition

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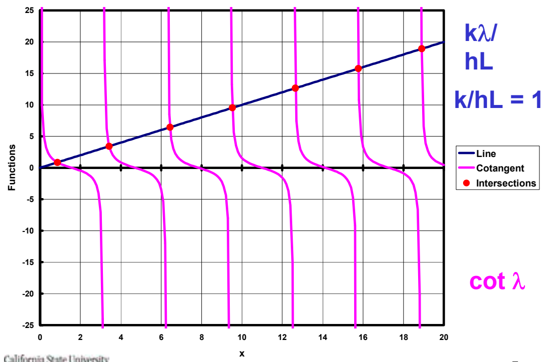
Review Finding λ_n



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Review Initial Conditions

- Usual formula for C_m but $\lambda_m \neq m\pi$

$$C_m = \frac{\int_0^1 \Theta_0(\xi)\cos(\lambda_m\xi)d\xi}{\int_0^1 \cos^2(\lambda_m\xi)d\xi} = \frac{2\lambda_m\int_0^1 \Theta_0(\xi)\cos(\lambda_m\xi)d\xi}{\cos(\lambda_m)\sin(\lambda_m)+\lambda_m}$$

- Constant initial temperature, T_0, gives $\Theta_0 = 1$

$$C_m = \frac{2\lambda_m\int_0^1 (1)\cos(\lambda_m\xi)d\xi}{\cos(\lambda_m)\sin(\lambda_m)+\lambda_m} = \frac{2\sin(\lambda_m)}{\cos(\lambda_m)\sin(\lambda_m)+\lambda_m}$$

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Review Solution

- General solution

$$\Theta(\xi,\tau) = \sum_{m=1}^{\infty} \underbrace{\frac{2\lambda_m\int_0^1 \Theta_0(\xi)\cos(\lambda_m\xi)d\xi}{\cos(\lambda_m)\sin(\lambda_m)+\lambda_m}}_{C_m} e^{-\lambda_m^2\tau}\cos(\lambda_m\xi)$$

- Solution for $T(x,0) = T_0$ ($\Theta_0 = 1$)

$$\Theta(\xi,\tau) = \sum_{m=1}^{\infty} \frac{2\sin(\lambda_m)e^{-\lambda_m^2\tau}\cos(\lambda_m\xi)}{\cos(\lambda_m)\sin(\lambda_m)+\lambda_m}$$

- Need root-finding method to obtain eigenvalues, $\lambda_m = (hL/k)\cot\lambda_m$

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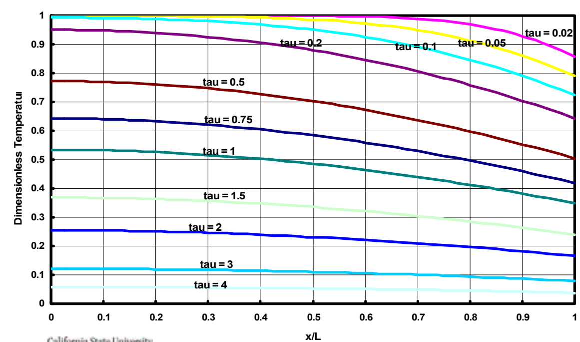
Review Solution for hL/k = 1



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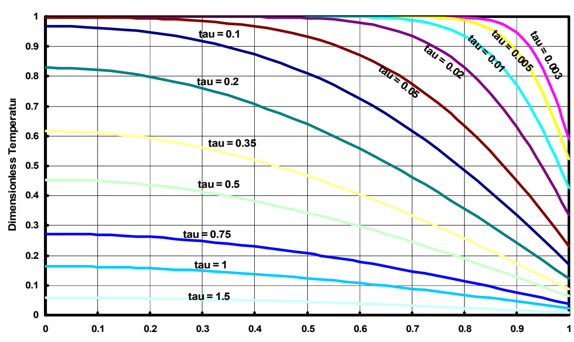
Review Solution for hL/k = 10



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Review Summary

- Create Sturm-Lioville problem for non-zero spatial boundary conditions
  - Define u(x,t) = v(x,t) + w(x)
  - Use $u - u_{ref}$ in original equation
- Solve by separation of variables
  - Time solution will be exponential
- Apply boundary conditions to determine eigenvalues
  - Also gets constants C_1 and C_2 which determines functions that are in solution

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Review Summary II

- Write solution as sum of all possible eigenfunctions with individual constants
- Use eigenfunction expansion to match initial conditions
  - If a solution for u(x,t) = v(x,t) + w(x) is used the eigenfunction expansion must be for $u_0(x) - w(x)$
- Solution is sum of all eigenfunctions with constants determined from matching initial conditions

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Cylindrical Diffusion Equation

- General diffusion equation for three dimensions $\frac{\partial u}{\partial t} = \alpha \nabla^2 u$

One-dimensional radial equation

$$\text{Cartesian} \quad \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\text{Cylindrical} \quad \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\text{Sphere} \quad \nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi}$$

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Radial Diffusion Equation

- Governs diffusion (heat conduction) in cylinder for $t \geq 0$ and $0 \leq r \leq R$
  - u(r,t) is temperature, species concentration
  - Initial condition $u(r,0) = u_0(r)$
  - No boundary condition at r = 0 except that u(0,t) is finite
    - $\partial u/\partial r|_{r=0} = 0$ by symmetry
- Diffusivity, α, is material property (length)^2/(time)

$$\frac{\partial u}{\partial t} = \alpha \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r}$$

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New Variable for u(R,t) = u_R

- $u(r,t) = v(r,t) + u_R$
  - v(r,t) satisfies diffusion equation
  - v(R,t) = 0 and v(0,t) is finite
    - Gives a Sturm-Liouville problem for radial function
- Since u_R is a constant, u(r,t) is a solution to the original problem
  - It satisfies the differential equation and the boundary conditions

$$\frac{\partial u}{\partial t} = \frac{\partial (v + u_R)}{\partial t} = \alpha \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} = \alpha \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial (v + u_R)}{\partial r}$$

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Separation of Variables

- Assume v(r,t) = P(r)T(t)

$$\frac{\partial u}{\partial t} = \frac{\partial [P(r)T(t)]}{\partial t} = P(r)\frac{\partial T(t)}{\partial t} =$$

$$\alpha \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} = \alpha \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial [P(r)T(t)]}{\partial r} = \alpha T(t) \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial P(r)}{\partial r}$$

- Divide by $\alpha P(r)T(t)$

$$\frac{1}{\alpha}\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = \frac{1}{P(r)}\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial P(r)}{\partial r} = -\lambda^2$$

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Solve ODEs to Get v(r,t)

- Have exponential ODEs in time and Bessel's equation for radial function

$$\frac{dT(t)}{dt} = -\lambda^2 \alpha T(t) \qquad T(t) = Ae^{-\lambda^2 \alpha t}$$

$$\frac{d}{dr} r \frac{dP(r)}{dr} + \lambda^2 r P(r) = 0$$

- This is form of Bessel's equation
- Solution of Bessel's equation is $P(r) = BJ_0(\lambda r) + CY_0(\lambda r)$
  - J_0 and Y_0 are (order zero) Bessel functions of first and second kind, respectively

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Review Bessel's Equation

$$\frac{d^2y(x)}{dx^2}+\frac{1}{x}\frac{dy(x)}{dx}+\frac{x^2-\nu^2}{x^2}y(x)=0$$

- Arises in mechanical and thermal problems in circular geometries
- The value of ν is a known parameter
- Use power series solution technique known as Frobenius method
- Have two linearly independent solutions

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Gamma Functions

- Function $\Gamma(x)$ generalizes factorials to non-integer arguments
- Definition $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$
- Analog of (n+1)! = (n+1)n! $\Gamma(x+1)=x\Gamma(x)$
- For integer x, $\Gamma(n+1) = n! = n\Gamma(n)$
- Used for coefficients of Bessel functions with noninteger order

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Bessel Functions, First Kind

- Separate expressions for integer and non-integer values of ν
- Use n for integer values of ν

$$J_\nu(x)=\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+\nu}}{2^{2m+\nu}m!\,\Gamma(m+\nu+1)} \Rightarrow J_n(x)=\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+n}}{2^{2m+n}m!\,(m+n)!}$$

- First few terms

$$J_n(x)=\left(\frac{x}{2}\right)^n\left[\frac{1}{n!}-\frac{1}{1!(n+1)!}\left(\frac{x}{2}\right)^2+\frac{1}{2!(n+2)!}\left(\frac{x}{2}\right)^4+\cdots\right]$$

- Plots for n = 0,1, and 4 on next chart

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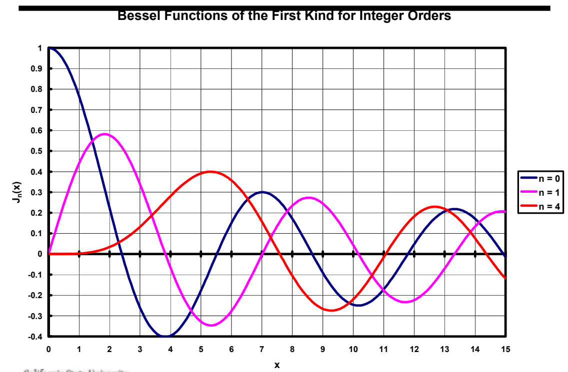
Bessel Function Plot



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Bessel Functions, Second Kind

- $Y_n(x)$ is defined as follows for integer $\nu = n$

$$Y_n(x)=\frac{2}{\pi}\left[J_n(x)\left(\ln\frac{x}{2}+\gamma\right)+x^n\sum_{m=0}^{\infty}\left\{\frac{(-1)^{m-1}x^{2m}}{2^{2m+n+2}m!(m+n)!}\left(\sum_{k=1}^{m}\frac{1}{k}+\sum_{k=1}^{m+n}\frac{1}{k}\right)\right\}+x^{-n}\sum_{m=0}^{n-1}\left\{\frac{(n-m-1)!\,x^{2m}}{2^{2m-n+2}m!(m+n)!}\right\}\right]$$

Not present in equation for Y_0

- General solution to Bessel's equation given by Bessel functions of order n or ν
  - y(x) = $AJ_n(x) + BY_n(x)$ for integer ν = n
  - y(x) = $AJ_\nu(x) + BJ_{-\nu}(x)$ for noninteger ν

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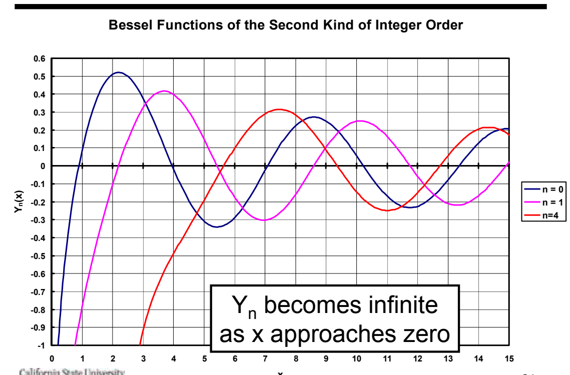
Bessel Function Plot



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Bessel Equation Solutions

- If we want a solution for x = 0 we cannot use $Y_n(x)$ so a general solution that includes x = 0 is $y(x) = AJ_n(x)$
- Formally define $Y_\nu(x)$ for non-integer ν

$$Y_\nu(x) = \frac{(\cos \nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$$

- In limit as n approaches an integer, this definition approaches $Y_n(x)$
- Gives $y(x) = AJ_\nu(x) + BY_\nu(x)$ for any ν

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Bessel Function Summary

- Bessel's equation, $x^2 d^2y/dx^2 + x dy/dx + (x^2 - \nu^2)y = 0$, main applications are to problems in radial geometries.
- Physical problem gives value for ν
- The general solution to Bessel's equation is $y = C_1 J_\nu(x) + C_2 Y_\nu(x)$ where C_1 and C_2 are constants that are determined by the boundary conditions on the differential equation.

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Bessel's Equation Summary II

- $J_\nu(x)$ and $Y_\nu(x)$: Bessel functions, order ν, first and second kind, respectively.
  - have oscillatory behavior
  - found in various tables and computer library solutions
  - At x = 0, $J_0(x) = 1$ and $J_n(x) = 0$ for $n \neq 0$
  - As x approaches zero, $Y_n(x)$ approaches minus infinity
- Can transform some equations into the form of Bessel's equation.

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Radial Diffusion Solution

- Transform Bessel's equation whose solution is $y = AJ_n(x) + BY_n(x)$
- Define z = x/k so $y = AJ_n(kz) + BY_n(kz)$
- Transformed equation is

$$\frac{d}{dz} z \frac{dy}{dz} + \left(-\frac{\nu^2}{z} + k^2 z \right) y = 0$$

- Radial diffusion equation has $\nu = 0$

$$\frac{d}{dr} r \frac{dP}{dr} + \lambda^2 r P = 0$$

- Solution is $P = AJ_0(\lambda r) + BY_0(\lambda r)$

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Transformation Details

- Bessel's equation $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2) y = 0$
- Define z = x/k so that x = kz
  - dy/dx = dy/d(kz) = (1/k)dy/dx
  - $d^2y/dx^2 = = d^2y/d(kz)^2 = (1/k^2) d^2y/dx^2$
  - Substitute into Bessel's equation and divide entire equation by z

$$\frac{1}{z}\left[(kz)^2 \frac{1}{k^2} \frac{d^2y}{dz^2} + kz \frac{1}{k} \frac{dy}{dz} + (k^2 z^2 - \nu^2) y \right]$$

$$z \frac{d^2y}{dz^2} + \frac{dy}{dz} + \left(-\frac{\nu^2}{z} + k^2 z \right) y = \frac{d}{dz} z \frac{dy}{dz} + \left(-\frac{\nu^2}{z} + k^2 z \right) y = 0$$

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Bessel as Sturm-Liouville

- Compare transformed Bessel's equation to Sturm-Liouville problem

Bessel's equation $\frac{d}{dz}\left[z \frac{dy}{dz} \right] + \left(-\frac{\nu^2}{z} + k^2 z \right) y = 0$

Sturm-Liouville $\frac{d}{dz}\left(r(z) \frac{dy}{dz} \right) + [q(z) + \lambda p(z)] y = 0$

- Bessel is Sturm-Liouville equation with r(z) = p(z) = z, $q(z) = -\nu^2/z$ and $\lambda = k^2$

Weight function p(z)

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Radial Diffusion Solution II

- Radial diffusion equation for P(r): $\frac{d}{dr} r \frac{dP(r)}{dr} + \left(\frac{0}{r} + \lambda^2 r\right) P(r) = 0$
- Transformed Bessel equation: $\frac{d}{dz}\left[z \frac{dy}{dz}\right] + \left(-\frac{\nu^2}{z} + k^2 z\right) y = 0$
- Solution to second (general ν) equation is y = $AJ_\nu(kz) + BY_\nu(kz)$
- Solution to radial diffusion (ν = 0) equation is $AJ_0(\lambda r) + BY_0(\lambda r)$

r is weight function

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Radial Diffusion Solution III

- Start with general solution P(r) = $AJ_0(\lambda r) + BY_0(\lambda r)$
- Solution applies in region to $0 \le r \le R$
- Must have P(r) finite at r = 0
- Since $Y_0(r) \to -\infty$ as $r \to 0$ we must have B = 0 for P(r) to be finite at r = 0
- Condition P(R) = 0 requires $J_0(\lambda R) = 0$
- Need solutions, α, of equation $J_0(\alpha) = 0$
  - Call solutions to $J_0(\alpha) = 0$ the zeros of J_0

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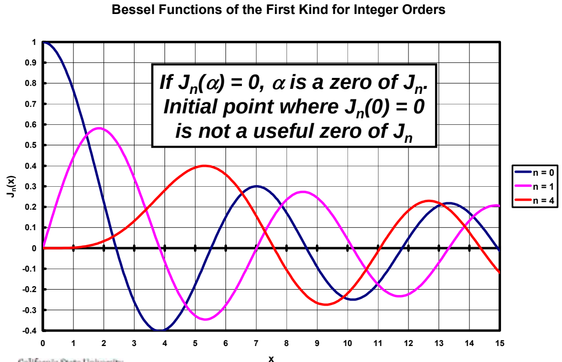
Zeros of Bessel Functions



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Bessel Function Zeros, α_{mn}

	$J_0(\alpha_{m0}) = 0$	$J_1(\alpha_{m1}) = 0$	$J_2(\alpha_{m2}) = 0$
m = 1	2.40483	3.83171	5.13562
m = 2	5.52008	7.01559	8.41724
m = 3	8.65373	10.17347	11.61984
m = 4	11.79153	13.32369	14.79595
m = 5	14.93092	16.47063	17.95982

Ref: Abramowitz and Stegun, NBS AMS 55

Note increase by about π between successive zeros

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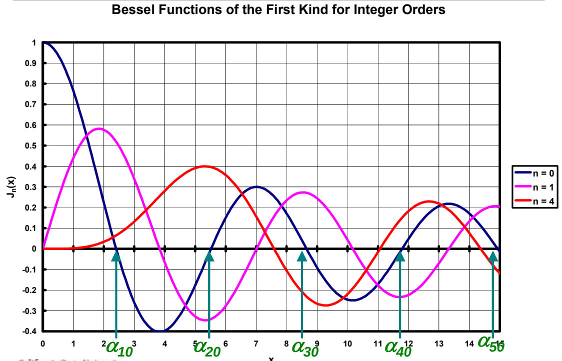
Zeros of J_0



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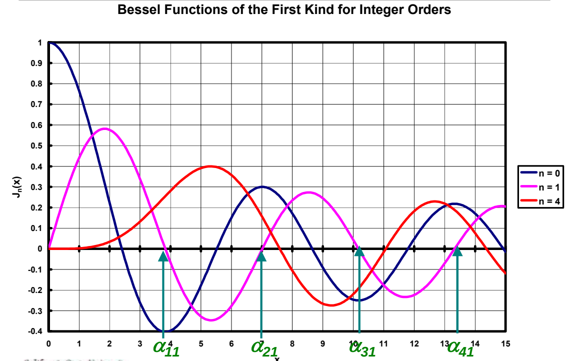
Zeros of J_1



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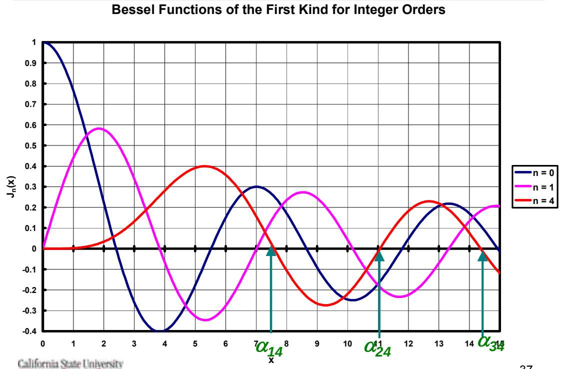
Zeros of J_4



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Bessel Eigenfunctions, $J_n(\alpha_{mn})$

- Different eigenfunctions are all Bessel functions of the same order, n
  - Unlike sines and cosines we can have many different Bessel functions
  - For both kinds of functions, the eigen-functions are given by the zeros
  - $\sin(n\pi) = 0$ or $\cos[(2n+1)\pi/2] = 0$
  - $J_n(\alpha_{mn}) = 0$ does not have fixed intervals like the sine and cosine
  - For radial solution, $J_0(\lambda R) = 0$ gives $\lambda_m R = \alpha_{m0}$; define $k_{m0} = \alpha_{m0}/R$, so $\lambda_m = k_{m0}$

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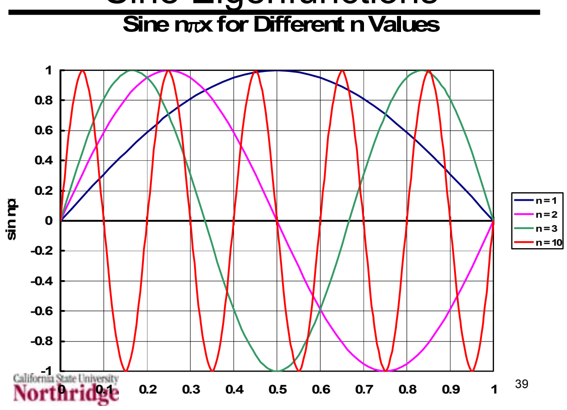
Sine Eigenfunctions



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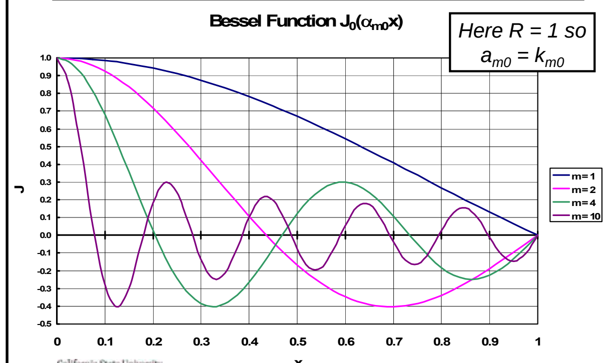
Bessel (J_0) Eigenfunctions



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Bessel Function Expansion

- General eigenfunction expansions $f(x) = \sum_{m=1}^{\infty} a_m y_m(x)$
- Bessel eigen-functions: $J_n(k_{mn}x)$ $f(x) = \sum_{m=1}^{\infty} a_m J_n(k_{mn}x)$

$$a_m = \frac{\int_a^b p(x) y_m f(x)dx}{\int_a^b p(x) y_m^2 dx} = \frac{\int_0^R x J_n(k_{mn}x) f(x)dx}{\int_0^R x[J_n(k_{mn}x)]^2 dx}$$

p(x) = x is weight function

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Normalization Integral

- Integral of $xJ_n(k_{mn}x)dx$ depends on n and definition of k_{mn}
- For n = 0 general result is

$$\int_0^R x[J_0(k_{mn}x)]^2 dx = \frac{R^2}{2}\left[J_0^2(k_{mn}R) + J_1^2(k_{mn}R)\right]$$

- For $J_0(k_{mn}R) = 0$ this simplifies to

$$\int_0^R x[J_0(k_{mn}x)]^2 dx = \frac{R^2}{2}\left[J_0^2(k_{mn}R) + J_1^2(k_{mn}R)\right]$$

- General result uses Struve H function

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Integrals to Compute a_m

- Integrals with Bessel functions may give complicated results
  – Use symbolic int function of MATLAB
- Some simple cases are possible using equations on page 194 of Kreyszig

$$\int x^{\nu} J_{\nu-1}(k_{m\nu-1}x)dx = \frac{x^{\nu} J_{\nu}(k_{m\nu-1}x)}{k_{m\nu-1}}$$

- Useful only when desired power of x matches the Bessel function order

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Example: f(x) = 1

- Construct Bessel function expansion for f(x) = 1 in region 0 ≤ x ≤ 1 using J_0 in case where $J_0(k_{m0}R) = 0$ for R = 1
- Here we have a "simple" integral with ν – 1 = 0 so ν = 1

$$a_m = \frac{2\int_0^R x J_0(k_{m0}x) f(x)dx}{[RJ_1(k_{m0}R)]^2} = \frac{2\int_0^1 \frac{x}{R} J_0\left(\frac{\alpha_{m0}}{R}x\right)(1) d\left(\frac{x}{R}\right)}{[J_1(\alpha_{m0})]^2} \qquad \xi = \frac{x}{R}$$

$$= \frac{2\left[\frac{\xi J_1(\alpha_{m0}\xi)}{\alpha_{m0}}\right]_{\xi=0}^{\xi=1}}{[J_1(\alpha_{m0})]^2} = \frac{2\left[\frac{J_1(\alpha_{m0})}{\alpha_{m0}} - 0\right]}{[J_1(\alpha_{m0})]^2} = \frac{2}{\alpha_{m0} J_1(\alpha_{m0})}$$

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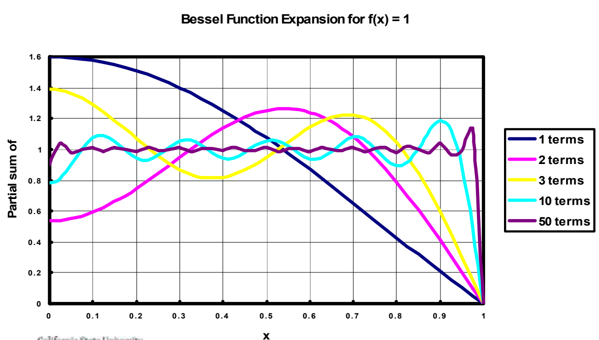
Example Expansion



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Back to Radial Diffusion Problem

- Have exponential ODEs in time and Bessel's equation for radial function

$$\frac{dT(t)}{dt} = -\lambda^2 \alpha T(t) \qquad \frac{d}{dr} r \frac{dP(r)}{dr} + \lambda^2 r P(r) = 0$$

$$v(r,t) = T(t)P(r) = Ae^{-\lambda^2 \alpha t}\left[BJ_0(\lambda r) + CY_0(\lambda r)\right]$$

- Boundary conditions: C = 0 and $\lambda_m R = \alpha_{m0}$

$$u(r,t) = \sum_{m=1}^{\infty} C_m e^{-\lambda_m^2 \alpha t} J_0(\lambda_m r) + u_R \qquad \lambda_m R = \alpha_{m0}$$

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Initial Condition

- Radial equation for P(r) is a Sturm-Liouville problem so we use eigenfunc-tion expansion for initial condition
  – Region is 0 ≤ r ≤ R and p(r) = r is weight function

$$u_0(r) = u(r,0) = \sum_{m=1}^{\infty} C_m J_0(\lambda_m r) + u_R \qquad \lambda_m R = \alpha_{m0}$$

$$C_m = \frac{\int_0^R r J_0(\lambda_m r)(u_0 - u_R)dr}{\int_0^R r[J_0(\lambda_m r)]^2 dr} = \frac{\int_0^R r J_0(\lambda_m r)(u_0 - u_R)dr}{\frac{R^2}{2}[J_1(\lambda_m R)]^2}$$

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Example: $u_0(r)$ = U, a Constant

$$C_m = \frac{\int_0^R r J_0(\lambda_m r)(U - u_R)dr}{\frac{R^2}{2}[J_1(\lambda_m R)]^2} = \frac{2(U-u_R)}{R^2} \frac{\left[\frac{rJ_1(\lambda_m r)}{\lambda_m}\right]_{r=0}^{r=R}}{[J_1(\lambda_m R)]^2}$$

$$= \frac{2(U-u_R)\frac{RJ_1(\lambda_m R)}{\lambda_m}}{R^2[J_1(\lambda_m R)]^2} = \frac{2(U-u_R)}{\lambda_m R J_1(\lambda_m R)} = \frac{2(U-u_R)}{\alpha_{m0} J_1(\alpha_{m0})}$$

$$u(r,t) = \sum_{m=1}^{\infty} \frac{2(U-u_R)}{\alpha_{m0} J_1(\alpha_{m0})} e^{-\alpha_{m0}^2 \frac{\alpha t}{R^2}} J_0\left(\alpha_{m0} \frac{r}{R}\right) + u_R$$

$$\frac{u - u_R}{U - u_R} = f\left(\frac{\alpha t}{R^2}, \frac{r}{R}\right)$$

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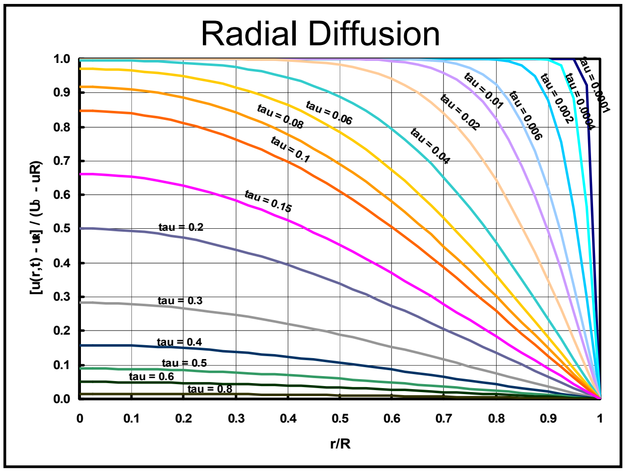
Radial Diffusion
[u(r,t) - uR] / (Ui - uR)
r/R
tau = 0.0001
tau = 0.0004
tau = 0.002
tau = 0.006
tau = 0.01
tau = 0.02
tau = 0.04
tau = 0.06
tau = 0.08
tau = 0.1
tau = 0.15
tau = 0.2
tau = 0.3
tau = 0.4
tau = 0.5
tau = 0.6
tau = 0.8