Unsteady Heat Transfer

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Mechanical Engineering 375

Heat Transfer

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California State University Northridge

Outline

- Review material on fins
- Lumped parameter model
  - Basis for and derivation of model
  - Solving lumped-parameter problems
- Unsteady solutions using charts
  - Differential equation as basis for charts
  - Problem solving with charts
  - Semi-infinite solutions
  - Product solutions

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Fin review

- Adds surface to enhance heat transfer
- Analysis for single fin linked to analysis of surface with multiple fins
- Equations for simple fins and charts for fin efficiency and effectiveness
- Rectangular fin equations: $m = (hp/kA_c)^{1/2}$

$$\frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L_c - x)}{\cosh mL_c} \qquad \frac{\dot{Q}_{fin}}{T_b - T_\infty} = \sqrt{kA_c hp}\tanh mL_c$$

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Lumped Parameter Model

- Simplified model of unsteady heat transfer for a particular problem
  - Solid object, with constant k and a uniform initial temperature, T_i
  - Placed in fluid environment with constant temperature, T_∞, at zero time (t = 0)
  - Convection to the solid with constant heat transfer coefficient, h
  - Under certain conditions the temperature in the solid is assumed uniform

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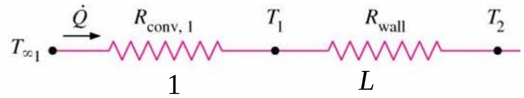
Parallel Resistance

- Look at solid object initially at an initial temperature, T_i, placed into a medium at another temperature T_∞ with heat transfer coefficient h
- Convective resistance is 1/hA, conductive resistance is L/kA

Figure 3-6) in Çengel, *Heat and Mass Transfer*



$$\frac{1}{hA} \qquad \frac{L}{kA}$$

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Uniform Temperature Basis

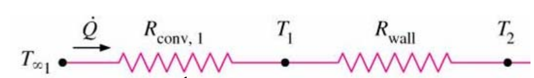

Figure 3-6) in Çengel, *Heat and Mass Transfer*

$$\frac{1}{hA} \qquad \frac{L}{kA}$$

$$\dot{Q} = \frac{T_{\infty 1} - T_1}{\frac{1}{hA}} = \frac{T_1 - T_2}{\frac{L}{kA}} \Rightarrow \frac{T_1 - T_2}{T_{\infty 1} - T_1} = \frac{\frac{L}{kA}}{\frac{1}{hA}} = \frac{hL}{k}$$

- $T_1 - T_2$ will be small compared to $T_{\infty 1} - T_1$ when hL/k is small

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Application to Unsteady Case

- Unsteady case: temperatures change
- Special case: convection resistance is much larger than conduction in solid
- Result: temperature differences in the solid are almost negligible
- Idealization: Assume that solid is at uniform temperature, T
- Model: $\rho c_p V dT/dt = hA(T_\infty - T)$
  - V is volume

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Lumped Parameter Model

- Basic model says that convection energy into solid $hA(T_\infty - T)$ goes to increase uniform solid temperature, T, giving energy change $\rho c_p V dT/dt$

$$\rho c_p V \frac{dT}{dt} = hA(T_\infty - T) \quad \Rightarrow \quad \frac{dT}{dt} = -\frac{hA}{\rho c_p V}(T - T_\infty)$$

- Define characteristic length, $L_c = V/A$, and inverse time constant, $b = hA/\rho c_p V = hA/\rho c_p L_c$

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Lumped Parameter Solution

- Have first-order differential equation with initial condition that $T = T_i$ at $t = 0$

$$\frac{dT}{dt} = -\frac{hA}{\rho c_p V}(T - T_\infty) = -b(T - T_\infty) \quad T = T_i \text{ at } t = 0$$

- Solution is known to be exponential

$$(T - T_\infty) = (T_i - T_\infty)e^{-bt} \quad or \quad T = (T_i - T_\infty)e^{-bt} + T_\infty$$

- You can show that solution satisfies differential equation and initial condition

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Lumped Parameter Analogy

- Combine solution with b definition

$$(T - T_\infty) = (T_i - T_\infty)e^{-bt} = (T_i - T_\infty)e^{-\frac{hAt}{\rho c_p V}}$$

- $1/hA$ is convection resistance and $\rho c_p V$ is capacity of solid to absorb heat

$$\frac{hAt}{\rho c_p V} = \frac{t}{\frac{1}{hA}\rho c_p V} = \frac{t}{R_{thermal}C_{thermal}}$$

- Result same as RC electrical circuit

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When can we use this?

- Saw that solid temperature becomes closer to uniform as hL/k (known as Biot number, Bi) becomes smaller
- Criterion for application of lumped parameter solution is $Bi = hL_c/k < 0.1$
- Problem: what is L_c for a cylinder with a diameter of 0.1 m and a length of 0.5 m?

$$L_c = \frac{V}{A} = \frac{\frac{\pi}{4}D^2L}{2\frac{\pi}{4}D^2 + \pi DL} = \frac{1}{\frac{2}{L} + \frac{4}{D}} = \frac{1}{\frac{2}{0.5\,m} + \frac{4}{0.1\,m}} = 0.02273\,m$$

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When can we use this? II

- So $L_c = 0.02273$ m; can we use lumped parameter model if $h = 80$ W/m²·K and $k = 240$ W/m²·K?
- Criterion for application of lumped parameter solution is $Bi = hL_c/k < 0.1$

$$Bi = \frac{hL_c}{k} = \frac{\frac{80\,W}{m^2 \cdot K}(0.02273\,m)}{\frac{240\,W}{m \cdot K}} = 0.0076 < 0.1$$

- So lumped parameter model is valid

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Application of the Model

- If the cylinder analyzed previously has c_p = 900 J/kg·K, ρ = 2700 kg/m^3, an initial temperature of 350^{O}C and an environmental temperature of 30^{o}C, how long will it take to reach 50^{o}C?
- **Given:** L_c = 0.02273 m, h = 80 W/m^2·K, c_p = 900 J/kg·K, ρ = 2700 kg/m^3, T_i = 350^{O}C, T_∞ = 30^{o}C, T = 50^{o}C
- **Find:** time, t, to reach T = 50^{o}C

Application of the Model II

- **Given:** h = 80 W/m^2·K, c_p = 900 J/kg·K, ρ = 2700 kg/m^3, T_i = 350^{O}C, T_∞ = 30^{o}C, T = 50^{o}C; **Find:** t

$$(T - T_\infty) = (T_i - T_\infty)e^{-bt} \quad \Rightarrow \quad t = -\frac{1}{b}\ln\left(\frac{T - T_\infty}{T_i - T_\infty}\right)$$

$$b = \frac{hA}{\rho c_p V} = \frac{h}{\rho c_p L_c} = \frac{\frac{80\,W}{m^2 \cdot K}\frac{1\,J}{W \cdot s}}{\frac{2700\,kg}{m^3}\frac{900\,J}{kg \cdot K}(0.02273\,m)} = \frac{0.001449}{s}$$

Application of the Model III

$$t = -\frac{1}{b}\ln\left(\frac{T - T_\infty}{T_i - T_\infty}\right) = -\frac{s}{0.001449}\ln\left(\frac{50^o C - 30^o C}{350^o C - 30^o C}\right) = 1914\,s$$

- Lumped parameter approach easy to use and does not require application of more complex calculations
- Applicable to general geometry
- Requires Bi = hL_c/k < 0.1
- Next consider unsteady problems with spatial variation

Review Conduction Equation

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} k \frac{\partial T}{\partial x} + \frac{\partial}{\partial y} k \frac{\partial T}{\partial y} + \frac{\partial}{\partial z} k \frac{\partial T}{\partial z} + \dot{e}_{gen}$$

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} k \frac{\partial T}{\partial x} + \frac{\partial}{\partial y} k \frac{\partial T}{\partial y} + \frac{\partial}{\partial z} k \frac{\partial T}{\partial z} + \dot{e}_{gen}$$

For constant k, bring k outside the derivative

0 for one dimensional heat transfer

0 for no heat generation

$$\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \quad \Rightarrow \quad \frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial x^2}$$

1D, constant k, $\dot{e}_{gen} = 0$

- Define the thermal diffusivity, α = $k/\rho c_p$
- Dimensions: $\frac{E}{T \cdot L \cdot \Theta}\frac{L^3}{M}\frac{M \cdot \Theta}{E} = \frac{L^2}{T}$
- Typical units are m^2/s or ft^2/s

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial x^2} \quad \Rightarrow \quad \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

- Solution requires initial (t=0, all x) and boundary conditions

What is 1D?

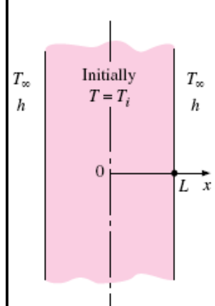

Figure 4-11(a) in Çengel, *Heat and Mass Transfer*

- In theory, one dimensional heat transfer implies that the slab is infinite (or insulated) in the y and z directions
- Practically this means that the y and z dimensions are so large that end effects are not important

Specific Problem

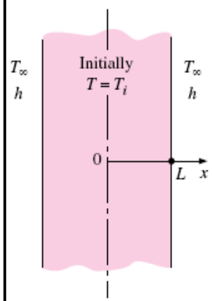

Figure 4-11(a) in Çengel, *Heat and Mass Transfer*

- **Problem:** at t = 0, a large slab initially at T_i is placed in a medium at temperature T_∞ with a heat transfer coefficient, h
- **Coordinates:** Choose x = 0 as center of slab (which runs from –L to L) for this symmetric problem

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Specific Problem II

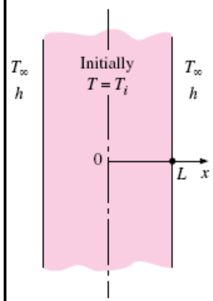

Figure 4-11(a) in Çengel, *Heat and Mass Transfer*

- **Initial condition:** at t = 0, T = T_i for all x
- **Boundary conditions:** At x = 0, $\partial T/\partial x = 0$ for symmetry. At x = L an energy balance gives $-k\partial T/\partial x = h(T - T_\infty)$
- **Dimensionless form:** shows important combinations of variables

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Specific Problem Conditions

- Differential equation with boundary and initial condition for T(x,t)

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad T(x,0) = T_i \qquad \left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h\left[T(L,t) - T_\infty\right]$$

- Define dimensionless distance, $\xi = x/L$ and dimensionless time, $\tau = \alpha t/L^2$, called the Fourier number

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Dimensionless Equation Form

- Define dimensionless temperature ratio $\Theta = \dfrac{T - T_\infty}{T_i - T_\infty}$
- Get dimensionless differential equation and initial/boundary conditions for $\Theta(\xi,\tau)$

$$\frac{\partial \Theta}{\partial \tau} = \frac{\partial^2 \Theta}{\partial \xi^2} \qquad \Theta(\xi,0) = 1 \qquad \left.\frac{\partial \Theta}{\partial \xi}\right|_{\xi=0} = 0$$

$$-\left.\frac{\partial \Theta}{\partial \xi}\right|_{\xi=1} = \frac{hL}{k}\Theta(1,\tau) = Bi\,\Theta(1,\tau)$$

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Dimensionless Result

$$\Theta = \frac{T - T_\infty}{T_i - T_\infty} \qquad \tau = \frac{\alpha t}{L^2} \qquad \xi = \frac{x}{L}$$

$$\frac{\partial \Theta}{\partial \tau} = \frac{\partial^2 \Theta}{\partial \xi^2} \qquad \Theta(\xi,0) = 1 \qquad \left.\frac{\partial \Theta}{\partial \xi}\right|_{\xi=0} = 0$$

$$-\left.\frac{\partial \Theta}{\partial \xi}\right|_{\xi=1} = \frac{hL}{k}\Theta(1,\tau) = Bi\,\Theta(1,\tau)$$

- We started with the following variables to solve for T: T_i, T_∞, x, L, t, α, h, k
- We now see that $T = (T_i - T_\infty)\Theta + T_\infty$, where Θ depends on ξ, τ, and Bi

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Equation Solution

- Solution is infinite series with an infinite set of dimensionless parameters λ_n that depends on Bi

$$\Theta = \sum_{n=1}^{\infty} \frac{4\sin\lambda_n}{2\lambda_n + \sin 2\lambda_n} e^{-\lambda_n^2 \tau} \cos\lambda_n \xi$$

- The values of λ_n are the roots of the equation $\lambda_n/Bi = \cot\lambda_n$
  – Next chart shows first seven λ_n values for Bi = 1

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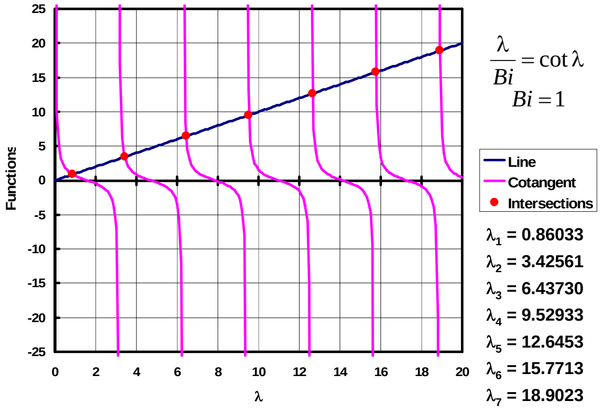
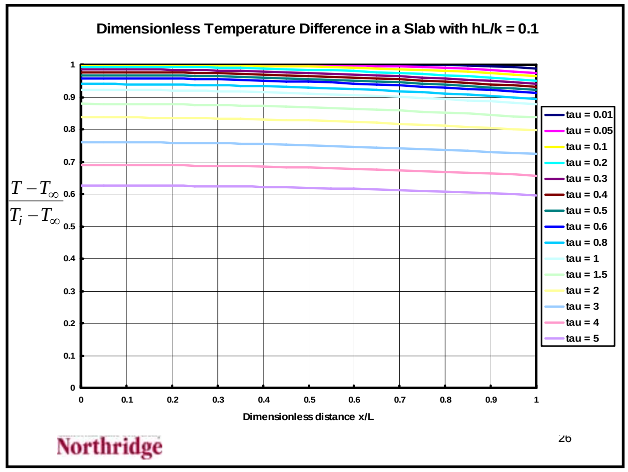
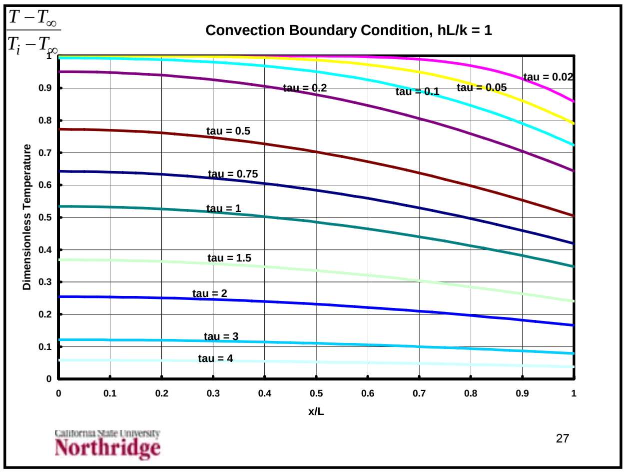
Finding λ

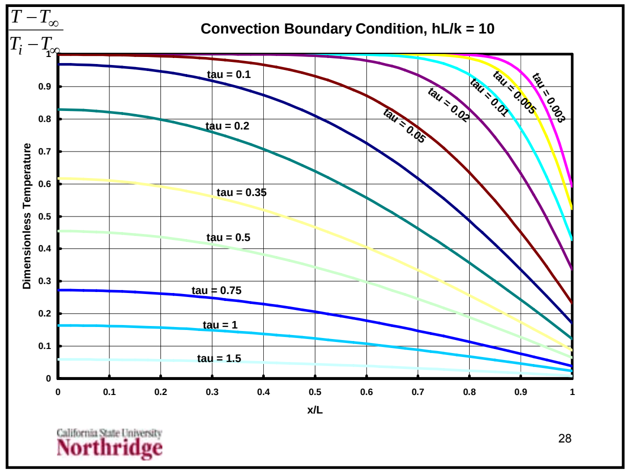


$$\frac{\lambda}{Bi} = \cot\lambda \qquad Bi = 1$$

$\lambda_1 = 0.86033$
$\lambda_2 = 3.42561$
$\lambda_3 = 6.43730$
$\lambda_4 = 9.52933$
$\lambda_5 = 12.6453$
$\lambda_6 = 15.7713$
$\lambda_7 = 18.9023$







Center-line Temperature, T_0

- Rewrite general result to introduce A_n

$$\Theta = \frac{T - T_\infty}{T_i - T_\infty} \sum_{n=1}^{\infty} \frac{4\sin\lambda_n}{2\lambda_n + \sin 2\lambda_n} e^{-\lambda_n^2\tau} \cos\lambda_n\xi = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2\tau} \cos\lambda_n\xi$$

- Result for center-line (x = 0)

$$\Theta_0 = \frac{T_0 - T_\infty}{T_i - T_\infty} = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2\tau} \cos\lambda_n 0 = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2\tau}$$

- Heisler charts show Θ_0 as a function of $\tau = \alpha t/L^2$ with $1/Bi = k/Lh$ as a parameter
  – A_n and λ_n depend on Bi

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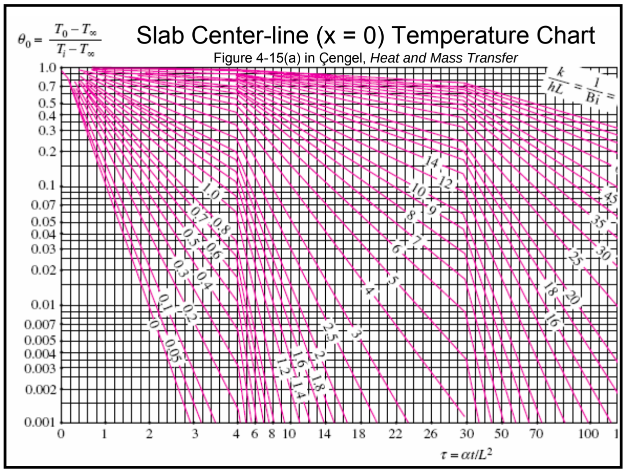
Slab Center-line (x = 0) Temperature Chart

Figure 4-15(a) in Çengel, *Heat and Mass Transfer*



Problem

- Find time required to cool the centerline temperature of 0.3 m thick plate to 50°C if k = 50 W/m·K, α = 15x10^{-6} m^2/s and initial temperature is 400°C. The heat transfer coefficient is 80 W/m^2·K and the environmental temperature = 20°C.
- **Given:** T_0 = 50°C, T_i = 400°C, T_∞ = 20°C, L = 0.3/2 = 0.15 m, k = 50 W/m·K, α = 15x10^{-6} m^2/s, h = 80 W/m^2·K **Find:** t

Solution

- Chart relates $(T_0 - T_\infty)/(T_i - T_\infty)$, k/hL, and $\alpha t/L^2$.
- From given data we can find

$$\frac{T_0 - T_\infty}{T_i - T_\infty} = \frac{50^oC - 20^oC}{400^oC - 20^oC} = 0.0789 \qquad \frac{k}{hL} = \frac{\frac{50\,W}{m\cdot K}}{\frac{80\,W}{m\cdot K}}\frac{1}{0.15\,m} = 4.167$$

- Find $\tau = \alpha t/L^2$ = 11.9 (see next chart)

$$t = \frac{\tau L^2}{\alpha} = \frac{11.9(0.15\,m)^2}{\frac{15x10^{-6}\,m^2}{s}} = 1.79x10^{-4}\,s = 4.96\ h$$

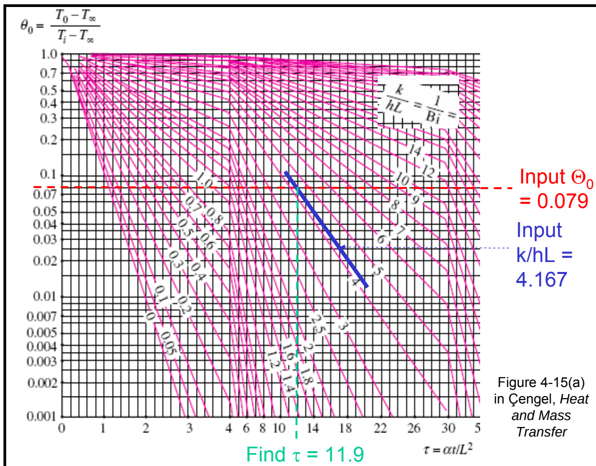
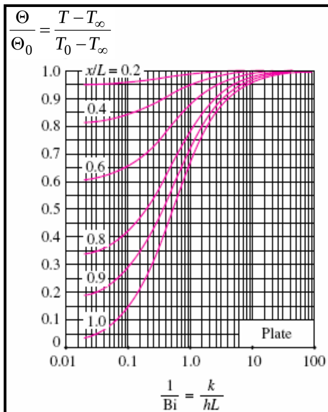


Figure 4-15(a) in Çengel, *Heat and Mass Transfer*

Chart II



Can find T at any x/L from this chart once T at x = 0 is found from previous chart

See basis for this chart on the next page

Figure 4-15(b) in Çengel, *Heat and Mass Transfer*

Temperature Approximations

- Previous equation for Θ/Θ_0

$$\frac{\Theta}{\Theta_0} = \frac{T - T_\infty}{T_0 - T_\infty} = \frac{A_1 e^{-\lambda_1^2\tau}\cos\lambda_1\xi + A_2 e^{-\lambda_2^2\tau}\cos\lambda_2\xi + \cdots}{A_1 e^{-\lambda_1^2\tau} + A_2 e^{-\lambda_2^2\tau} + \cdots}$$

- For "large" τ (> 0.2) series in numerator and denominator converge in one term

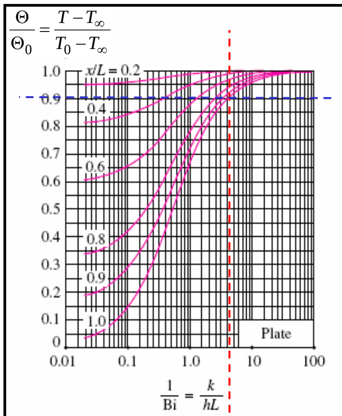
$$\frac{\Theta}{\Theta_0} = \frac{T - T_\infty}{T_0 - T_\infty} = \frac{A_1 e^{-\lambda_1^2\tau}\cos\lambda_1\xi + \cancel{A_2 e^{-\lambda_2^2\tau}\cos\lambda_2\xi + \cdots}}{A_1 e^{-\lambda_1^2\tau} + \cancel{A_2 e^{-\lambda_2^2\tau} + \cdots}} \approx \cos\lambda_1\xi$$

Temperature Approximations II

- For τ > 0.2

$$\frac{\Theta}{\Theta_0} = \frac{T - T_\infty}{T_0 - T_\infty} \approx \cos\lambda_1\xi$$

- Depends only on ξ = x/L and λ_1 which depends on Bi = hL/k
- Must first determine Θ_0 as in previous example to get Θ
- What is Θ at x = L in that example?



Answer

Example had T_0 = 50°C, T_∞ = 20°C, and k/hL = 4.167

This chart gives $(T - T_\infty)/(T_0 - T_\infty)$ = 0.91 for x/L = 1

So T = (0.91)·(50°C – 20°C) + 20°C = 47.3°C

Figure 4-15(b) in Çengel, *Heat and Mass Transfer*

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Alternative Approximation

- Return to series for "large" τ (> 0.2) that converges in one term

$$\Theta = \frac{T - T_\infty}{T_i - T_\infty} = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 \tau} \cos \lambda_n \xi \approx A_1 e^{-\lambda_1^2 \tau} \cos \lambda_1 \xi$$

- Can use this approximation to find Θ and can also solve for τ and ξ when τ > 0.2

$$\tau \approx -\frac{1}{\lambda_1^2} \ln\left(\frac{T - T_\infty}{T_i - T_\infty} \frac{1}{A_1 \cos \lambda_1 \xi} \right) \qquad \xi \approx \frac{1}{\lambda_1} \cos^{-1}\left(\frac{T - T_\infty}{T_i - T_\infty} \frac{1}{A_1 e^{-\lambda_1^2 \tau}} \right)$$

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Bi	λ_1	A_1
0.01	0.0998	1.0017
0.02	0.1410	1.0033
0.04	0.1987	1.0066
0.06	0.2425	1.0098
0.08	0.2791	1.0130
0.1	0.3111	1.0161
0.2	0.4328	1.0311
0.3	0.5218	1.0450
0.4	0.5932	1.0580
0.5	0.6533	1.0701
0.6	0.7051	1.0814
0.7	0.7506	1.0918
0.8	0.7910	1.1016
0.9	0.8274	1.1107
1.0	0.8603	1.1191

Table Extract

- Use Table 4-2 in text to find λ_1 and A_1 for given Bi
- Find T at any ξ = x/L and $\tau = \alpha t/L^2$ from

$$\Theta = \frac{T - T_\infty}{T_i - T_\infty} \approx A_1 e^{-\lambda_1^2 \tau} \cos \lambda_1 \xi$$

$$T \approx (T_i - T_\infty)\left(A_1 e^{-\lambda_1^2 \tau} \cos \lambda_1 \xi \right) + T_\infty$$

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Approximate Equations

- Repeat example with T_0 = 50°C, T_i = 400°C, L = 0.3/2 = 0.15 m, T_∞ = 20°C, k = 50 W/m·K, α = 15x10^{-6} m²/s, h = 80 W/m²·K to find t for T_0 = 50°C and T at surface for this time
- Recall 1/Bi = 4.167 so Bi = 0.24
- Interpolate in table to find λ_1 = 0.4684 and A_1 = 1.0367 for Bi = 0.24
  - Last two slides have interpolation details
- First find time for T_0 = 50°C

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Approximate Equations II

- Data: T_0 = 50°C, T_i = 400°C, L =0.15 m, T_∞ = 20°C, k = 50 W/m·K, α = 15x10^{-6} m²/s, h = 80 W/m²·K, λ_1 = 0.4476 and A_1 = 1.0334

$$\tau \approx -\frac{1}{\lambda_1^2} \ln\left(\frac{T - T_\infty}{T_i - T_\infty} \frac{1}{A_1 \cos \lambda_1 \xi} \right) = -\frac{1}{0.4684^2} \bullet$$

$$\ln\left(\frac{50^oC - 20^oC}{400^oC - 20^oC} \frac{1}{(1.0367)\cos 0} \right) = 11.74$$

$$t = \frac{\tau L^2}{\alpha} = \frac{11.74(0.15\,m^2)}{\frac{15x10^{-6}\,m^2}{s}} = 1.76x10^4\ s = 4.83\ h$$

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Approximate Equations III

- Find surface temperature for τ = 11.74

$$T = T_\infty + (T_i - T_\infty)\Theta \approx T_\infty + (T_i - T_\infty)\left(A_1 e^{-\lambda_1^2 \tau} \cos \lambda_1 \xi \right) = 20^oC +$$

$$= (400^oC - 20^oC)\left\{ 1.0334 e^{-0.4684^2 (11.74)} \cos[(0.4684)(1)] \right\} = 46.8^oC$$

- Results similar to charts
  - 4.83 h *vs.* 4.96 h to reach T_0 = 50°C
  - Surface temperature 46.8°C *vs.* 47.3°C
- For full solution T_0 = 50°C in 4.83 h and surface temperature = 46.7°C at that time

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Cylinder and Sphere

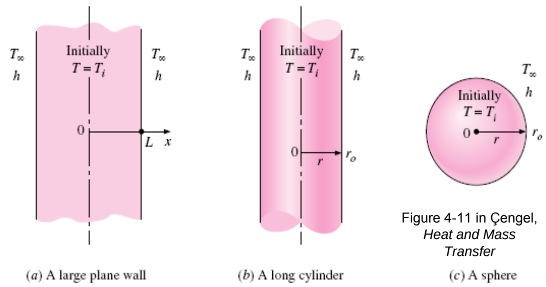

Figure 4-11 in Çengel, *Heat and Mass Transfer*

Same problem has similar chart solutions

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1D Cylinder, constant k, $\dot{e}_{gen} = 0$

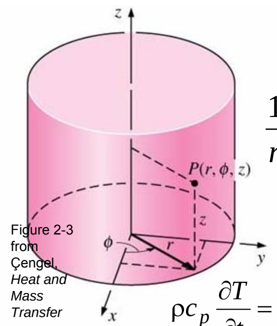

Figure 2-3 from Çengel, *Heat and Mass Transfer*

$$\rho c_p \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}kr\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \phi}k\frac{\partial T}{\partial \phi} + \frac{\partial}{\partial z}k\frac{\partial T}{\partial z} + \dot{e}_{gen}$$

$$\rho c_p \frac{\partial T}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}r\frac{\partial T}{\partial r} \qquad \frac{\partial T}{\partial t} = \frac{\alpha}{r}\frac{\partial}{\partial r}r\frac{\partial T}{\partial r}$$

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Same Problem for Cylinder

- Constant initial temperature, T_i, and convection at outer radius r_0

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r}\frac{\partial}{\partial r}r\frac{\partial T}{\partial r} \qquad T(r,0) = T_i \qquad \left.\frac{\partial T}{\partial r}\right|_{r=0} = 0$$

$$-k\left.\frac{\partial T}{\partial r}\right|_{r=r_0} = h\left[T(r_0,t) - T_\infty\right]$$

- Define dimensionless distance, $\xi = r/r_0$ and dimensionless time, $\tau = \alpha t/r_0^2$,

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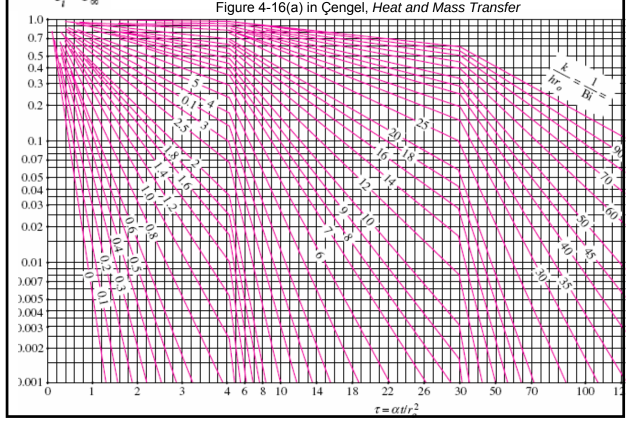
$\theta_0 = \frac{T_0 - T_\infty}{T_i - T_\infty}$ Cylinder Center (r = 0) Temperature Chart

Figure 4-16(a) in Çengel, *Heat and Mass Transfer*



1D Sphere, constant k, $\dot{e}_{gen} = 0$

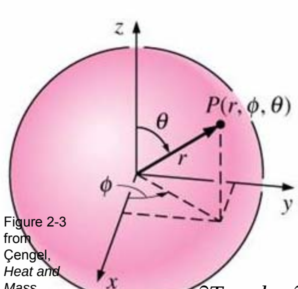


Figure 2-3 from Çengel, *Heat and Mass Transfer*

$$\rho c_p \frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}kr^2\frac{\partial T}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}k\sin\theta\frac{\partial T}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}k\frac{\partial T}{\partial \phi} + \dot{e}_{gen}$$

$$\rho c_p \frac{\partial T}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial T}{\partial r} \qquad \frac{\partial T}{\partial t} = \frac{\alpha}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial T}{\partial r}$$

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Same Problem for Sphere

- Constant initial temperature, T_i, and convection at outer radius r_0

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial T}{\partial r} \qquad T(r,0) = T_i \qquad \left.\frac{\partial T}{\partial r}\right|_{r=0} = 0$$

$$-k\left.\frac{\partial T}{\partial r}\right|_{r=r_0} = h\left[T(r_0,t) - T_\infty\right]$$

- Define dimensionless distance, $\xi = r/r_0$ and dimensionless time, $\tau = \alpha t/r_0^2$,

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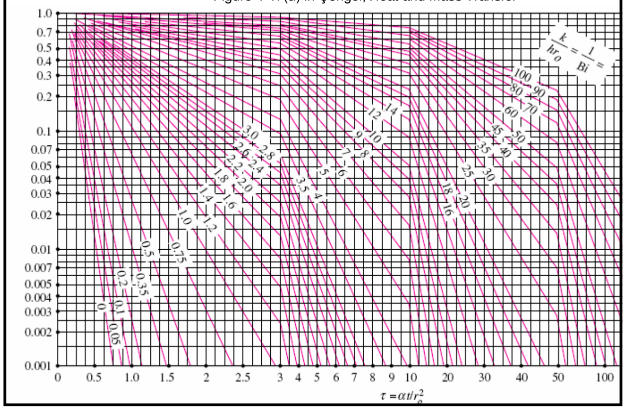
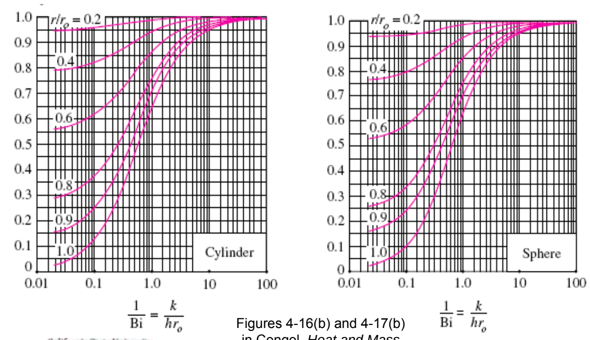
$\theta_0 = \frac{T_0 - T_\infty}{T_i - T_\infty}$

Sphere Center (r = 0) Temperature Chart

Figure 4-17(a) in Çengel, *Heat and Mass Transfer*



(T – T∞)/(T0 – T∞) Charts



Figures 4-16(b) and 4-17(b) in Çengel, *Heat and Mass Transfer*

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More Approximate Solutions

- Cylinder and sphere also have approximate solutions for τ > 0.2
  - Values of A_1 and λ_1 still depend on Bi and are different for each geometry
- Cylinder $\Theta = \frac{T - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} J_0\left(\lambda_1 \frac{r}{r_0}\right)$
- Sphere $\Theta = \frac{T - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \frac{r_0}{\lambda_1 r} \sin\left(\lambda_1 \frac{r}{r_0}\right)$

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More Approximate Solutions II

Coefficients used in the one-term approximate solution of transient one-dimensional heat conduction in plane walls, cylinders, and spheres (Bi = *hL/k* for a plane wall of thickness 2*L*, and Bi = hr_o/k for a cylinder or sphere of radius r_o)

	Plane Wall		Cylinder		Sphere	
Bi	λ_1	A_1	λ_1	A_1	λ_1	A_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.1	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.2	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.3	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880

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Table 4-2 in Çengel, *Heat and Mass Transfer*

More Approximate Solutions III

- The function $J_0(x)$ is the zero order Bessel function
  - Tables in text or get value from Excel function besselj(x,0) or Matlab function besselj(0,x)
    - Excel function requires analysis tool pack add-in to be installed
- Here Bi = hr_0/k for cylinder and sphere is different from Bi = $hV/(kA)$ for lumped parameter solution

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Semi-Infinite Solids

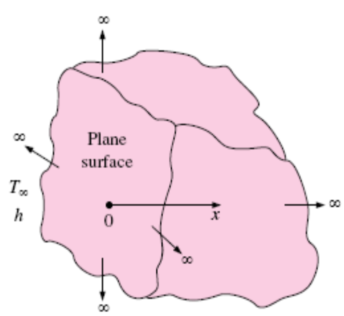

Figure 4-24 in Çengel, *Heat and Mass Transfer*

Plane that extends to infinity in all directions

Practical applications: large area for short times

- Example: earth surface locally

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Two Semi-infinite Solid Results

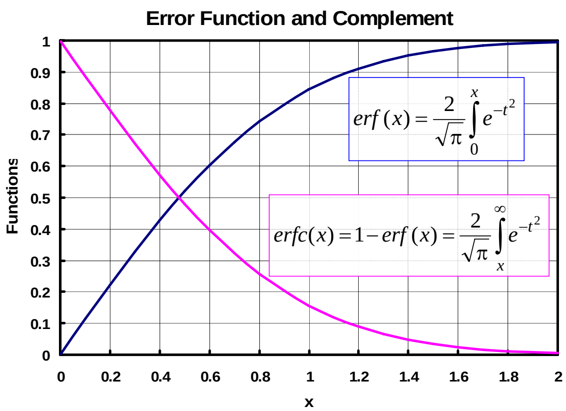
- Initial temp T_i, surface temperature set to T_s at t = 0 $$\frac{T-T_s}{T_i-T_s}=erf\left(\frac{x}{\sqrt{4\alpha t}}\right)$$
- Initial temp T_i, convection starts at t = 0 to T_∞ with heat transfer coefficient, h

$$\frac{T-T_i}{T_\infty-T_i}=erf\left(\frac{x}{\sqrt{4\alpha t}}\right)-e^{\left(\frac{hx}{k}+\frac{h^2\alpha t}{k^2}\right)}erfc\left(\frac{x}{\sqrt{4\alpha t}}+\frac{h\sqrt{\alpha t}}{k}\right)$$

Error Functions

- Defined integrals erf(x) is called error function and erfc(x) = 1 – erf(x) is the complementary error function
  - Excel/ Matlab functions erf(x) and erfc(x)
    - Excel requires analysis took pack add-in
- See plots on next chart

$$erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} \qquad erfc(x)=1-erf(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}$$



Semi-Infinite Medium Results

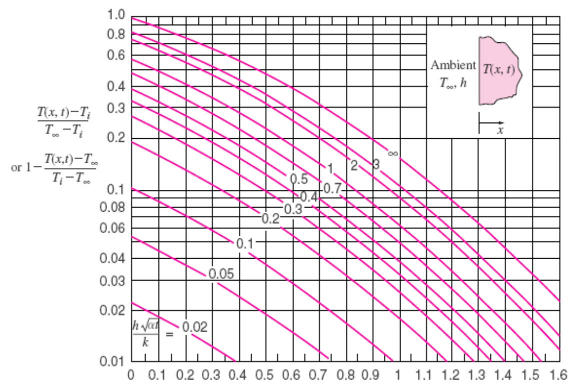


Figure 4-29 in Çengel, *Heat and Mass Transfer* $\eta=\frac{x}{2\sqrt{\alpha t}}$

Problem

- How deep should water pipes be buried in a soil that is initially at 20°C to avoid freezing if the surface is at –15°C for 60 days? (Assume soil α = 1.4x10^{-7} m^2/s)
- **Given:** Soil with α = 1.4x10^{-7} m^2/s and T_i = 20°C must not freeze for t = 60 days if T_s= -15°C. **Find:** depth required
- **Assumption:** Required depth will set T = 0°C at t = 60 days

Solution

- Use equation for semi-infinite medium with fixed surface temperature

$$\frac{T-T_s}{T_i-T_s}=\frac{0^oC-(-15^oC)}{20^oC-(-15^oC)}=0.42857=erf\left(\frac{x}{\sqrt{4\alpha t}}\right)$$

- From tables or erfinv function of Matlab find erf^{-1}(.42857) = 0.40019 = $x/\sqrt{4\alpha t}$

$$x=0.40019\sqrt{\alpha t}=0.40019\sqrt{\frac{1.4x10^{-7}m^2}{s}(60\,d)\frac{86400\,s}{d}}$$

x = 0.682 m

Multidimensional Solutions

- Can get multidimensional solutions as product of one dimensional solutions
  - All one-dimensional solutions have initial temperature, T_i, with convection coefficient, h, and environmental temperature, T_∞, starting at t = 0
  - General rule: $\Theta_{twoD} = \Theta_{one}\Theta_{two}$ where Θ_{one} and Θ_{two} are solutions from charts for plane, cylinder or sphere

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Multidimensional Example

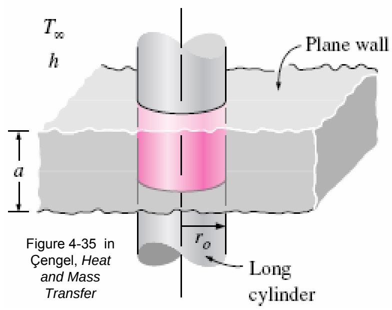

Figure 4-35 in Çengel, *Heat and Mass Transfer*

- Solution for finite cylinder is product of solution for infinite cylinder and infinite slab
- Slab solutions have thickness 2L so we use L = a/2 in this case

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Multidimensional Example II

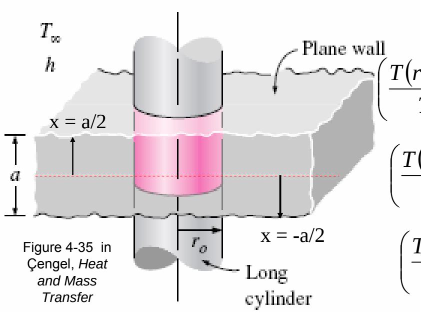

Figure 4-35 in Çengel, *Heat and Mass Transfer*

$$\left(\frac{T(r,x,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{finite\\cylinder}} = \left(\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{infinite\\cylinder}} \bullet \left(\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{infinite\\slab}}$$

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Multidimensional Problem

- A cylinder is 0.3 m high, with a radius of 0.1 m, k = 50 W/m·K, $\alpha = 15\text{x}10^{-6}$ m²/s, and initial temperature of 400°C. The heat transfer coefficient is 80 W/m²·K and the environmental temperature is 20°C. Find temperature in the centre of the cylinder and at its corner after one hour.

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Multidimensional Solution

- We can apply the following equation:

$$\left(\frac{T(r,x,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{finite\\cylinder}} = \left(\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{infinite\\cylinder}} \left(\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{infinite\\slab}}$$

- First get the temperatures at the center

$$\left(\frac{T(0,0,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{finite\\cylinder}} = \left(\frac{T(0,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{infinite\\cylinder}} \left(\frac{T(0,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{infinite\\slab}}$$

- We find infinite quantities from charts

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Multidimensional Solution II

- We have separate Biot and Fourier numbers for the two infinite geometries

$$\frac{k}{hr_0} = \frac{\frac{50\,W}{m\cdot K}}{\frac{80\,W}{m\cdot K}}\frac{1}{0.1\,m} = 6.25 \qquad \tau = \frac{\alpha t}{r_0^2} = \frac{\frac{15x10^{-6}m^2}{s}(3600\,s)}{(0.1\,m)^2} = 5.40$$

$$\frac{k}{hL} = \frac{\frac{50\,W}{m\cdot K}}{\frac{80\,W}{m\cdot K}}\frac{1}{0.15\,m} = 4.167 \qquad \tau = \frac{\alpha t}{L^2} = \frac{\frac{15x10^{-6}m^2}{s}(3600\,s)}{(0.15\,m)^2} = 2.40$$

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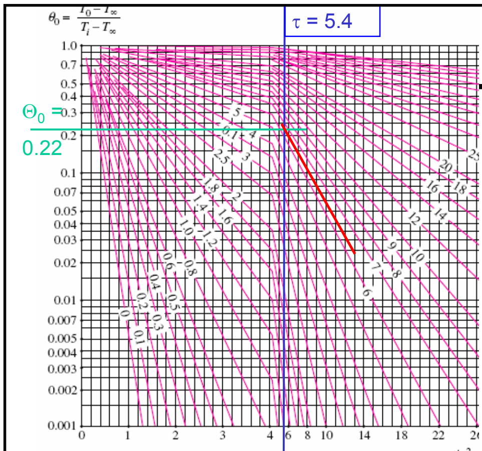

Figure 4-16(a) in Çengel, *Heat and Mass Transfer*

Enter **cylinder chart** with values of τ = 5.4 and k/hr$_0$ = 6.25 to find Θ_0 = 0.22

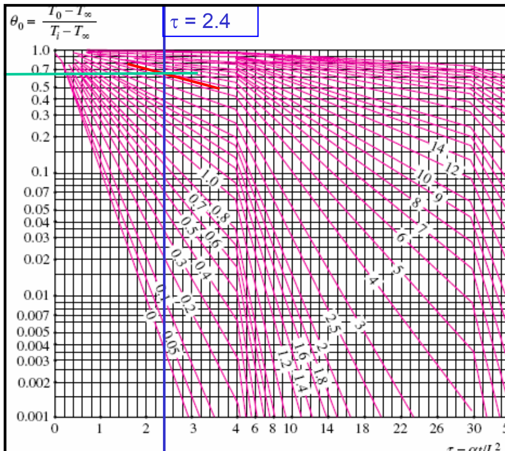

Figure 4-15(a) in Çengel, *Heat and Mass Transfer*

Slab chart uses τ = 2.4 and k/hL = 4.167 to find Θ_0 = 0.62

Multidimensional Solution III

- Use Θ_0 values just found

$$\left(\frac{T(0,0,t)-T_\infty}{T_i - T_\infty}\right)_{\substack{finite\\cylinder}} = \left(\frac{T(0,t)-T_\infty}{T_i - T_\infty}\right)_{\substack{infinite\\cylinder}} \left(\frac{T(0,t)-T_\infty}{T_i - T_\infty}\right)_{\substack{infinite\\slab}}$$

$$\left(\frac{T(0,0,3600\ s)-T_\infty}{T_i - T_\infty}\right)_{\substack{finite\\cylinder}} = (0.22)(0.62) = 0.1364$$

$$T_0 = T(0,0,3600\ s) = T_\infty + (T_i - T_\infty)(0.1364) = 20^oC + (400^oC - 20^oC)(0.1364) = \boxed{72^oC}$$

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Multidimensional Solution IV

- Edge temperature (x = L = 0.15 m and r = r$_0$ = 0.1 m) found as follows

$$\left(\frac{T(L,r_0,t)-T_\infty}{T_i - T_\infty}\right)_{\substack{finite\\cylinder}} = \left(\frac{T(r_0,t)-T_\infty}{T_i - T_\infty}\right)_{\substack{infinite\\cylinder}} \left(\frac{T(L,t)-T_\infty}{T_i - T_\infty}\right)_{\substack{infinite\\slab}}$$

- Each term on right is product of center solution times ratio of point to center
  - Must find this for each one dimensional solution separately
    - Use auxiliary charts for this ratio

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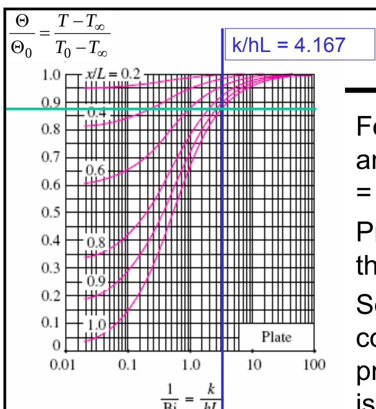

Slab

For k/hL = 4.167 and x/L = 1, Θ/Θ_0 = 0.875

Previously found that Θ_0 = 0.62

So slab component of product solution is (0.62)(0.875) = 0.543

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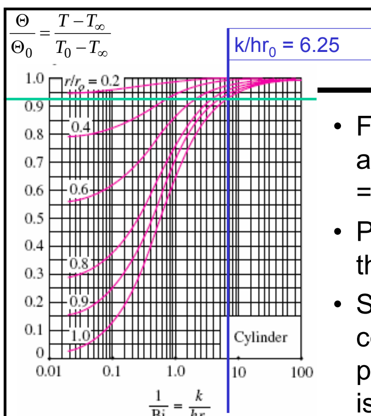
Figure 4-15(b) in Çengel, *Heat and Mass Transfer*



Cylinder

- For k/hr$_0$ = 6.25 and r/r$_0$ = 1, Θ/Θ_0 = 0.93
- Previously found that Θ_0 = 0.22
- So cylinder component of product solution is (0.22)(0.93) = 0.205

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Figure 4-15(b) in Çengel, *Heat and Mass Transfer*

Multidimensional Solution V

$$\left(\frac{T(L,r_0,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{finite\\cylinder}} = \left(\frac{T(r_0,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{infinite\\cylinder}} \left(\frac{T(L,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{infinite\\slab}}$$

$$\left(\frac{T(L,r_0,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{finite\\cylinder}} = (0.205)(0.543) = 0.111$$

$$T_{edge} = T(L,r_0,3600\ s) = T_\infty + (T_i - T_\infty)(0.111) =$$
$$20^oC + (400^oC - 20^oC)(0.111) = \boxed{62^oC}$$

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Repeat Using One-term Values

$$\left(\frac{T(L,r_0,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{finite\\cylinder}} = \left(\frac{T(r_0,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{infinite\\cylinder}} \left(\frac{T(L,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{infinite\\slab}}$$

$$\left(\frac{T(L,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{infinite\\slab}} = A_1 e^{-\lambda_1^2\tau}\cos\lambda_1\frac{x}{L}$$

$$\left(\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{infinite\\cylinder}} = A_1 e^{-\lambda_1^2\tau} J_0\left(\lambda_1\frac{r}{r_0}\right)$$

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Repeat Using One-term Values II

For infinite slab, τ = 2.40 and Bi = 0.24; interpolation in Table 4-2 for Bi = 0.24 gives λ_1 = 0.4684 and A_1 = 1.0367

$$\left(\frac{T(L,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{infinite\\slab}} = A_1 e^{-\lambda_1^2\tau}\cos\lambda_1\frac{x}{L}$$

$$= 1.0367 e^{-(0.4684)^2(2.40)} \cos\left(0.4684\frac{0.15\ m}{0.15\ m}\right) = 0.546$$

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Repeat Using One-term Values III

For infinite cylinder, τ = 5.40 and Bi = 0.16; interpolation in Table 4-2 for Bi = 0.24 gives λ_1 = 0.5469 and A_1 = 1.0388

$$\left(\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{infinite\\cylinder}} = A_1 e^{-\lambda_1^2\tau} J_0\left(\lambda_1\frac{r}{r_0}\right)$$

$$= 1.0388 e^{-(0.5469)^2(5.40)} J_0\left(0.5469\frac{0.1\ m}{0.1\ m}\right) = 0.191$$

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Repeat Using One-term Values IV

Now have individual terms in product solution

$$\left(\frac{T(L,r_0,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{finite\\cylinder}} = \left(\frac{T(r_0,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{infinite\\cylinder}} \left(\frac{T(L,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{infinite\\slab}}$$
$$= (0.546)(0.191) = 0.1045$$

$$T_{edge} = T(L,r_0,3600\ s) = T_\infty + (T_i - T_\infty)(0.1045) =$$
$$20^oC + (400^oC - 20^oC)(0.1045) = 60^oC$$

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Interpolation

- Given table with two data pairs (x_1,y_1) and (x_2,y_2) we want to find y for some value of x between x_1 and x_2

$$y = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) = y_1 + \frac{x - x_1}{x_2 - x_1}(y_2 - y_1)$$

- Here known x is Bi and we are trying to find y = λ_1 so interpolation formula is

$$\lambda_1 = \lambda_{1,1} + \frac{\lambda_{1,2} - \lambda_{1,1}}{Bi_2 - Bi_1}(Bi - Bi_1)$$

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Bi	λ_1
0.01	0.0998
0.02	0.1410
0.04	0.1987
0.06	0.2425
0.08	0.2791
0.1	0.3111
0.2	0.4328
0.3	0.5218
0.4	0.5932
0.5	0.6533
0.6	0.7051
0.7	0.7506
0.8	0.7910

Interpolation II

- Find λ_1 for Bi = 0.24
  - Bi = 0.24 is between Bi_1 = 0.2 ($\lambda_{1,1}$ = .4328) and Bi_2 = 0.3 ($\lambda_{1,2}$ = .5218)

$$\lambda_1 = \lambda_{1,1} + \frac{\lambda_{1,2} - \lambda_{1,1}}{Bi_2 - Bi_1}(Bi - Bi_1)$$

$$= .4328 + \frac{.5218 - .4328}{.3 - .2}(.24 - .2)$$

$$= .4684$$

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