Solving Simultaneous **Nonlinear** Algebraic Equations

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Numerical Analysis of Engineering Systems

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Outline

- Problem Definition of solving simultaneous nonlinear algebraic equations (SNAE)
- Using the MATLAB `fsolve` function
- Using Excel Solver
- Excel Solver exercise
- Newton's Method (Newton-Raphson procedure) for solving SNAE

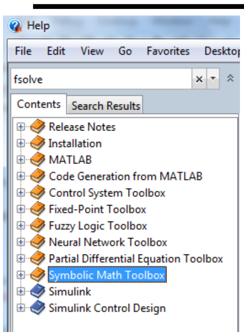
Generalize Simple Example

- Two nonlinear equations in two variables
  $2x - y - e^{-x} = 0$
  $-x + 2y - e^{-y} = 0$
- How do we generalize this
  - Have N equations in N unknowns
  - Unknowns are labeled $x_1, x_2, \ldots, x_N$
  - Each equation is written in the form $f_k = 0$
  - In general, $f_k = f_k(x_1, x_2, \ldots, x_N) = 0$
    - Many equations will not depend on all variables, but in general any equation could do so

Problem Definition

- We consider a system of N equations in the N variables $x_1, x_2, \ldots, x_N$
  - The vector variable, **x**, represents the set of unknowns $[x_1, x_2, \ldots, x_N]$
- The N equations have the form $f_1(\mathbf{x}) = 0$, $f_2(\mathbf{x}) = 0$, …, $f_N(\mathbf{x}) = 0$
  - We can use the vector representation $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, for the system of equations
    - Vector is $\mathbf{f} = [f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_N(\mathbf{x})]$
- We want to find the solution, $\mathbf{x} = [x_1, x_2, \ldots, x_N]$ to this set of equations

MATLAB Components



- MATLAB has a basic system plus additional components called toolboxes
  - Extract from help window shows toolboxes on computers in JD 1126
    - Optimization toolbox, with `function fsolve` is not available

MATLAB Function `fsolve`

- This MATLAB function to solve $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ is the multidimensional analog of `fzero`
- It has a similar function call
  - <Results> = fsolve(<function>, <initial guess>, <options>)
    - <function> returns the vector **f**(**x**)
    - <initial guess> is the vector guess $\mathbf{x}_0$
  - Look at example
    - $f_1 = 2x_1 - x_2 - e^{-x_1} = 0$
    - $f_2 = -x_1 + 2\,x_2 - e^{-x_2} = 0$

Using `fsolve`

- Write the vector function to compute the values of the two equations f_1 and f_2

```
function F = testFct(x)
    F = …
[2*x(1) - x(2) - exp(-x(1)); …
-x(1) + 2*x(2) - exp(-x(2))];
end    %not required
```

- Save this function file as testFct.m somewhere on your MATLAB path

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Using `fsolve` II

- Define a for matrix for the initial guesses of x_1 and x_2: `guess = [-5 -5]`
- Call the `fsolve` function

```
>> fsolve(@testFct, guess)
```

- MATLAB uses the default variable, ans

```
ans = -0.5671  -0.5671
```

- Test result by substituting into function

```
>> fval = testFct(ans)
fval = -0.4059e-6 -0.4059e-6
```

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More on `fsolve`

- Third parameter, <options> used to set calculation options
  - Can set parameters such as convergence tolerance, iteration display, algorithm used
    - Typically use a routine, optimset, to set options
    - opt=optimset('Display','iter'); %Sets option to display result of each iteration
- fsolve has multiple outputs including, solution, function values, codes, *etc.*
  - [x f flag out] = fsolve(@myFunc, guess, opt)

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Using Excel Solver

- Solver is a tool that varies selected cells to set a single cell to a fixed value a maximum value or a minimum value
- Main use is for constrained optimum (maximum or minimum problems)
- Can also be used for system of equations $f_k(\mathbf{x}) = 0$ as shown on following slides
  - To do this create cells for all x_k and all f_k then a single cell to compute $\sum_k f_k^2$

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Using Excel Solver II

- Solver is a tool that varies selected cells to set a single cell to a fixed value a maximum value or a minimum value
- Main use is for constrained optimum (maximum or minimum problems)
- Can also be used for system of equations $f_k(\mathbf{x}) = 0$ as follows
  - Write each equation in a separate cell
  - Write one cell formula to compute $\sum_k f_k^2$
  - Set cell with this formula equal to zero

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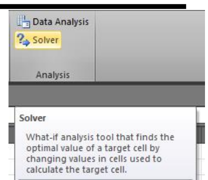
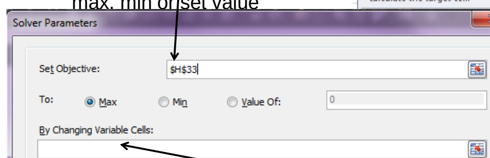
Using Solver

- Access from Analysis Group on Data tab
- Solver dialog box shown below
  - Objective cell is single cell for max, min or set value





Cells to be changed listed in this text box

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Solver Details for **f**(**x**) = **0**

- Enter each x_k value in a cell (usually in a single column)
  - Can use range names like x1_ or x_1
- Write a formula for each equation, $f_k(\mathbf{x})$ = 0 in a cell (typically a single column)
- Create objection function cell with formula to obtain $\sum_k f_k^2$
- Use solver to set objective function cell to zero by changing all x_k cells

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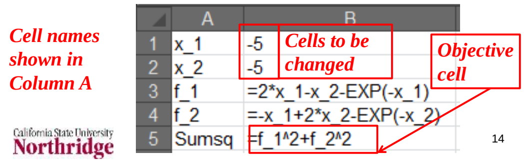
Solver on `fsolve` Example

- Use previous example from `fsolve`:
  - $f_1 = 2x_1 - x_2 - e^{-x_1} = 0$
  - $f_2 = -x_1 + 2\,x_2 - e^{-x_2} = 0$
- With same initial guess $x_1 = x_2 = -5$
  - Worksheet with initial conditions shown below in equation view



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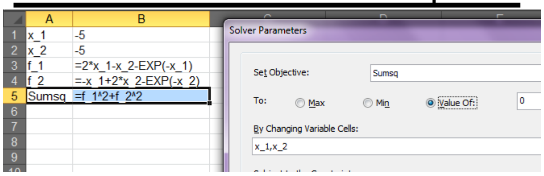
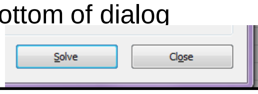
Solver Entries for Example



- Enter changing cells (with names x_1 and x_2) and objective cell (Sumsq)
- Select "Value Of:" and enter 0
- Click Solve at bottom of dialog



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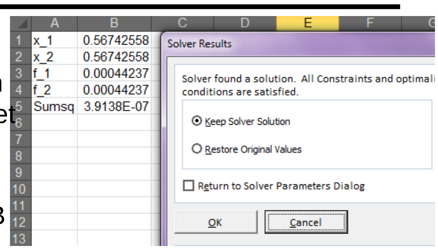
Solver Result

- Results dialog on worksheet
- Solution shown in column B



- Click OK to accept solution or Cancel to return to original values
- If no result obtained retry with new initial x_k

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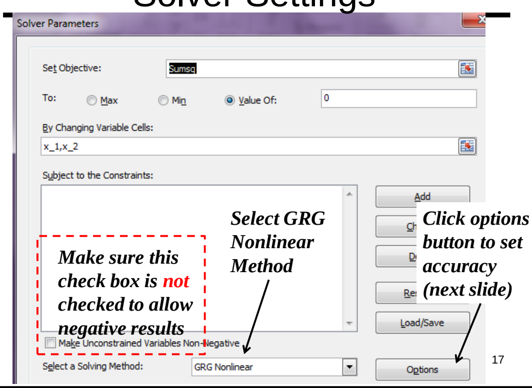
Solver Settings



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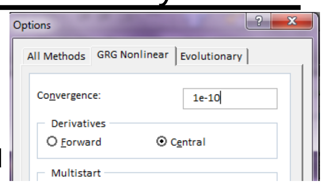
Solver Accuracy

- Options dialog appears after Options button (previous slide) is clicked



- Set small number in Convergence box to give good accuracy in solution
- Click OK at bottom of Options dialog to exit that dialog

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Your Solver Exercise

- Use Solver to find the intersection of a straight line and a circle in the x-y plane
  - Line: y = mx + i
  - Circle: $(x - a)^2 + (y - b)^2 = r^2$
  - Use m = 1, i = –1, a = b = 1, r = 2
  - Write as $f_k(x,y) = 0$, k = 1, 2
  - $f_1 = y - mx - i = 0$
  - $f_2 = (x - a)^2 + (y - b)^2 - r^2 = 0$
  - Solver may not be able to find both solutions, but try different guesses

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Newton's Method for **f**(**x**) = **0**

- Use notation $x_k^{(m)}$ and $f_k^{(m)}$ for values of x_k and f_k at iteration m
- Write Taylor series for $f_k^{(m+1)}$ in terms of values at previous iteration: $x_k^{(m)}$, $f_k^{(m)}$
  - Use only first-order terms
- $f_k^{(m+1)} = f_k^{(m)} + \sum_j \frac{\partial f_k}{\partial x_j}\left[x_j^{(m+1)} - x_j^{(m)}\right] + \cdots$
  - Set $f_k^{(m+1)} = 0$ for all f_k to get simultaneous linear algebraic equations: **Ay** = **b**

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How is this **Ay** = **b**

- $-f_k^{(m)} = \sum_j \frac{\partial f_k}{\partial x_j}\left[x_j^{(m+1)} - x_j^{(m)}\right] +$ *Ignore higher-order terms*
- $a_{kj} = \frac{\partial f_k}{\partial x_j}$ $\quad b_k = -f_k^{(m)}$ and $y_j = x_j^{(m+1)} - x_j^{(m)}$
- Partial derivative evaluated with values from iteration m
- After y_j are found we have new set of x iterations as $x_j^{(m+1)} = y_j + x_j^{(m)}$

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Return to `fsolve` Example

- Use previous example from `fsolve`:

$$f_1 = 2x_1 - x_2 - e^{-x_1} = 0$$
$$f_2 = -x_1 + 2\,x_2 - e^{-x_2} = 0$$
$$\frac{\partial f_1}{\partial x_1} = 2 + e^{-x_1} \qquad \frac{\partial f_1}{\partial x_2} = -1$$
$$\frac{\partial f_2}{\partial x_1} = -1 \qquad \frac{\partial f_2}{\partial x_2} = 2 + e^{-x_2}$$

- For each iteration we have to compute the **A** matrix of partial derivatives and the **b** vector of $-f_j$, and solve **Ay** = **b** for the y_j and get $x_j^{(m+1)} = y_j + x_j^{(m)}$

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Newton-Raphson Iteration Results

Each set of two rows has pattern shown in header

See NR workbook online

Iteration	x1 / x2	f1 / f2	df1/dx1 / df2/dx1	df1/dx2 / df2/dx2	y1 / y2
1	-5	-153.413	150.4132	-1	1.026771
	-5	-153.413	-1	150.4132	1.026771
2	-3.97323	-57.1291	55.15587	-1	1.054901
	-3.97323	-57.1291	-1	55.15587	1.054901
3	-2.91833	-21.4286	20.5103	-1	1.098324
	-2.91833	-21.4286	-1	20.5103	1.098324
4	-1.82	-7.99188	8.17188	-1	1.114336
	-1.82	-7.99188	-1	8.17188	1.114336
5	-0.70567	-2.73087	4.025198	-1	0.902706
	-0.70567	-2.73087	-1	4.025198	0.902706
6	0.197039	-0.62412	2.821159	-1	0.342705
	0.197039	-0.62412	-1	2.821159	0.342705
7	0.539744	-0.04315	2.582898	-1	0.027263
	0.539744	-0.04315	-1	2.582898	0.027263
8	0.567006	-0.00021	2.567221	-1	0.000137
	0.567006	-0.00021	-1	2.567221	0.000137
9	0.567143	-5.3E-09	2.567143	-1	3.4E-09
	0.567143	-5.3E-09	-1	2.567143	3.4E-09
10	0.567143	0	2.567143	-1	0
	0.567143	0	-1	2.567143	0

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Solution Hint

- Using Gaussian elimination on an exam or quiz is likely to have errors
  - You are more likely to get a better grade if you gave some simple indication of the operations you are doing
  - Here is an example

w – 3x + 5y +2z = 7
x + 6y – z = –2
2w + 5x –3y + 4z = 3
7w – 2x + y + z = 17

1	-3	5	2	7	Row 1
0	1	6	-1	-2	Row 2 – (0/1)(Row 1)
0	11	-13	0	-11	Row 3 – (2/1)(Row 1)
0	19	-34	-13	-32	Row 4 – (7/1)(Row 1)

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25

Quiz five

$$w + 2x - y + 3z = 5 \qquad w + 2x - y + 3z = -5$$
$$2w - 3x + 5y - 2z = 9 \qquad 2w - 3x + 5y - 2z = 20$$
$$4w + 4x - 2y + z = 13 \qquad 4w + 4x - 2y + z = 4$$
$$5w - 8x + 11y - 12z = 16 \qquad 5w - 8x + 11y - 12z = 60$$

Write two sets of equations above as augmented matrix at right and take first step below

1	2	−1	3	5	−5
2	−3	5	−2	9	20
4	4	−2	1	13	4
5	−8	11	−12	16	60

1	2	-1	3	5	-5
$2-2(1)$	$-3-2(2)$	$5-2(-1)$	$-2-2(3)$	$9-2(5)$	$20-2(-5)$
$4-4(1)$	$4-4(2)$	$-2-4(-1)$	$1-4(3)$	$13-4(5)$	$4-4(-5)$
$5-5(1)$	$-8-5(2)$	$11-5(-1)$	$-12-5(3)$	$16-5(5)$	$60-5(-5)$

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26

Quiz five II

Results of first row as pivot row at right and use of second row as pivot row shown below

1	2	−1	3	5	−5
0	−7	7	−8	−1	30
0	−4	2	−11	−7	24
0	−18	16	−27	−9	85

1	2	-1	3	5	-5
0	-7	7	-8	-1	30
0	$-4-(4/7)(-\mathbf{7})$	$2-(4/7)(\mathbf{7})$	$-11-(4/7)(-\mathbf{8})$	$-7-(4/7)(-\mathbf{1})$	$24-(4/7)(3$
0	$-\mathbf{1}8-(\frac{\mathbf{18}}{\mathbf{7}})(2)$	$11-5(-1)$	$-12-5(3)$	$16-5(5)$	$60-5(\mathbf{30}$

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27

Quiz Five Solution II

- After row 2 as pivot row

1	-3	5	2	7	Row 1
0	1	6	-1	-2	Row 2
0	0	-79	11	11	Row 3 – (11/1)(Row 2)
0	0	-148	6	6	Row 4 – (19/1)(Row 2)

Using row 3 as pivot subtract -148/(-79) times row 3 from row to give

0 0 0 -14.6076 -14.6076 Row 4

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28

Backsubstitution

w + –3x + 5y + 2z = 7
x + 6y + –1z = –2
-79y + 11z = 11
–14.6076z = –14.6076

z = –14.6076/(–14.6076) = 1

y = (11 – 11z)/(–79) = [11 – 11(1)]/(-79) = 0

x = [-2 + z – 6y]/1 = [–2 + 1 -6(0)]/1 = –1

w = [7 – 2z – 5y + 3x]/1 = [7 – 2(1) – 5(0) + 3(–1)]/1 = 2

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29