5 Invariant Measures

The Game of Life has been studied for $26+$ years. Nevertheless, most stable states of the infinite system remain a mystery. In a similar vein, the ergodic classifications of many of the LtL rules continue to be an enigma. However, the exploration of invariant configurations and invariant measures enables us to say something rigorous about the infinite systems of even the most nonlinear rules. In this chapter we present some of the results of our explorations along these lines.

5.1 Bounds and applications

Percolating through the Internet as electronic folklore were e-mail correspondences among John Conway, Dean Hickerson, and Hartmut Holzwart. The messages discussed a couple of bounds for the Game of Life. The first two theorems of this section are in the spirit of the electronic folklore, but formulated mathematically in terms of measures and generalized to the LtL family of rules.

Let ξ_0 be product measure with density p. Running the deterministic CA rule ξ_t on this random initial state yields a stochastic process, with updates determined by $(\rho, \beta_1, \beta_2, \delta_1, \delta_2)$. ξ_t may be thought of as a Markov process since the sites update independently from all preceding times except the current one. The Markov process is degenerate since the transitions are deterministic. Nevertheless, it has a compact state space, $\{0,1\}^{\mathbb{Z}^2}$, so there exists a measure μ that is invariant under the rule. (See [Lig], Theorem 1.8f.) Since the dynamics are translation invariant, μ can be chosen so.

Theorem 5.1.1. Let μ be a translation invariant measure for the LtL rule, ξ_t, which is determined by $(\rho, \beta_1, \beta_2, \delta_1, \delta_2)$. Let P_μ be the probability measure induced by μ. Let $p = \mu(\xi(x)=1) = P_\mu(\xi_0(x)=1)$ and $q = \mu(\xi(x)=0) = 1-p$. Then

$$p \leq \frac{4\rho(\rho+1)}{8\rho(\rho+1)-M}, \text{ where } M = \max\{\beta_2, \delta_2 - 1\}.$$

Proof. Since ξ_t is translation invariant and $\mathcal{N}$ is symmetric,

$$\sum_{y \in x+\mathcal{N}} P_\mu(\xi_0(x)=0, \xi_1(y)=1) = \sum_{x \in y+\mathcal{N}} P_\mu(\xi_0(x)=0, \xi_1(y)=1).$$

We compute an upper bound for the left-hand side of the above:

$$\sum_{y\in x+\mathcal{N}} P_\mu(\xi_0(x)=0,\xi_1(y)=1) \le 4\rho(\rho+1)P_\mu(\xi_0(x)=0) = 4\rho(\rho+1)(1-p).$$

The following will be used to obtain a lower bound for the right-hand side:

$$\begin{aligned}(1)\ P_\mu(\xi_0(x)=0,\xi_1(y)=1) &= P_\mu(\xi_0(x)=0\,|\,\xi_1(y)=1)P_\mu(\xi_1(y)=1)\\ &= P_\mu(\xi_1(y)=1)[1-P_\mu(\xi_0(x)=1\,|\,\xi_1(y)=1)]\\ &= P_\mu(\xi_1(y)=1)-P_\mu(\xi_0(x)=1,\ \xi_1(y)=1).\end{aligned}$$

$$\begin{aligned}(2)\ P_\mu(\xi_0(x)=1,\xi_1(y)=1) &= P_\mu(\xi_0(x)=1,\xi_1(y)=1,\xi_0(y)=0)\\ &\quad+P_\mu(\xi_0(x)=1,\xi_1(y)=1,\xi_0(y)=1).\end{aligned}$$

(3) For $a=0,1$,

$$P_\mu(\xi_0(x)=1,\xi_1(y)=1,\xi_0(y)=a) = E(1_{\xi_0(x)=1}\cdot 1_{\xi_1(y)=1}\cdot 1_{\xi_0(y)=a}).$$

$$\begin{aligned}(4)\ \sum_{x\in y+\mathcal{N}} E(1_{\xi_0(x)=1}\cdot 1_{\xi_1(y)=1}\cdot 1_{\xi_0(y)=a}) &= E(\sum_{x\in y+\mathcal{N}} 1_{\xi_0(x)=1}\cdot 1_{\xi_1(y)=1}\cdot 1_{\xi_0(y)=a})\\ &\le E(\ M\cdot 1_{\xi_1(y)=1}\cdot 1_{\xi_0(y)=a}).\end{aligned}$$

(The inequality in (4) holds because if $a=0$, then since y is a 0 at time 0 and a 1 at time 1, it sees at most β_2 $1's$ at time 0. If $a=1$, then since y remains a 1 at time 1, it sees at most δ_2-1 $1's$ at time 0.)

$$(5)\ E(\ 1_{\xi_1(y)=1}\cdot 1_{\xi_0(y)=0})+E(\ 1_{\xi_1(y)=1}\cdot 1_{\xi_0(y)=1}) = E(\ 1_{\xi_1(y)=1}) = P_\mu(\xi_1(y)=1).$$

Using the above in the order they appear yields:

$$\sum_{x\in y+\mathcal{N}} P_\mu(\xi_0(x)=0,\xi_1(y)=1)=\sum_{x\in y+\mathcal{N}} [P_\mu(\xi_1(y)=1)-P_\mu(\xi_0(x)=1,\,\xi_1(y)=1)]$$

$$=4\rho(\rho+1)p-\sum_{x\in y+\mathcal{N}} [P_\mu(\xi_0(x)=1,\xi_1(y)=1,\xi_0(y)=0)+P_\mu(\xi_0(x)=1,\xi_1(y)=1,\xi_0(y)=1)]$$

$$\geq 4\rho(\rho+1)p-M[E(\ 1_{\xi_1(y)=1}\cdot 1_{\xi_0(y)=0})+E(\ 1_{\xi_1(y)=1}\cdot 1_{\xi_0(y)=1})]$$

$$=4\rho(\rho+1)p-Mp.$$

Combining this bound with the upper bound attained above yields:

$$4\rho(\rho+1)p-Mp\leq 4\rho(\rho+1)(1-p)$$

and hence the desired inequality. □

What does Theorem 5.1.1 say about the invariant measures of specific rules? Let us mention two examples. It says that the density p of any translation invariant measure for the Game of Life satisfies, $p\leq 8/13$. For the range 2 LtL rule $(2,4,4,5,5)$ it says that $p\leq 6/11$.

Theorem 5.1.1 obtains an upper bound on the density of an invariant measure, μ. What about measures for which the time average densities of any trajectory of the rule are constant? In other words, can we find an upper bound on the density of a fixed, or *still life measure*? The answer is yes, and we prove it in Theorem 5.1.2; first let us define a *still life measure*.

Definition 5.1.1. A *still life measure* is a fixed measure μ. That is, starting from μ, the dynamics remain fixed for all time: $\xi_t^\mu=\xi_0^\mu$ for all t.

Theorem 5.1.2. Let μ be a still life measure. Let $p=\mu(\xi(x)=1)=P_\mu(\xi_0(x)=1)$ and $q=\mu(\xi(x)=0)=1-p$. Then

$$p\leq \frac{\sigma}{4\rho(\rho+1)-(\delta_2-1)+\sigma},$$

where $\sigma=$ the maximum number of live neighbors x can have at time $t=0$ when $\xi_0(x)=0$. σ depends on δ_2 since it is determined by computing the maximum number of live sites that can co-exist in $x+\mathcal{N}$ without overcrowding one another.

Proof. Let $x,y\in\Lambda$. Then,

$$\sum_{y\in x+\mathcal{N}} P_\mu(\xi_0(y)=1)=4\rho(\rho+1)p.$$

Also,

$$\sum_{y\in x+\mathcal{N}} P_\mu(\xi_0(y)=1)$$

$$=\sum_{y\in x+\mathcal{N}} [P_\mu(\xi_0(y)=1,\xi_1(x)=1)+P_\mu(\xi_0(y)=1,\xi_1(x)=0)]$$

$$=\sum_{y\in x+\mathcal{N}} [E(1_{\xi_0(y)=1}\cdot 1_{\xi_1(x)=1})+E(1_{\xi_0(y)=1}\cdot 1_{\xi_1(x)=0})]$$

$$=E(\sum_{y\in x+\mathcal{N}} 1_{\xi_0(y)=1}\cdot 1_{\xi_1(x)=1})+E(\sum_{y\in x+\mathcal{N}} 1_{\xi_0(y)=1}\cdot 1_{\xi_1(x)=0})$$

$$\leq E((\delta_2-1)1_{\xi_1(x)=1})+E(\sigma\cdot 1_{\xi_1(x)=0})=(\delta_2-1)p+\sigma(1-p).$$

Combining this bound with the equality on the first line yields:

$$4\rho(\rho+1)p\leq(\delta_2-1)p+\sigma(1-p)$$

and hence the desired inequality. □

Theorem 5.2.1 says that the density, p of any still life measure for the Game of Life satisfies $p\leq\frac{6}{11}$ (since $\delta_2=4$ implies that $\sigma=6$, see Appendix I). For the range 2 LtL rule $(2,4,4,5,5)$ it says that $p\leq\frac{3}{8}$ (since $\delta_2=5$ implies that $\sigma=12$, again see Appendix I).

Note: For all p in Theorem 5.1.2

$$p\leq\frac{4\rho(\rho+1)}{8\rho(\rho+1)-(\delta_2-1)},$$

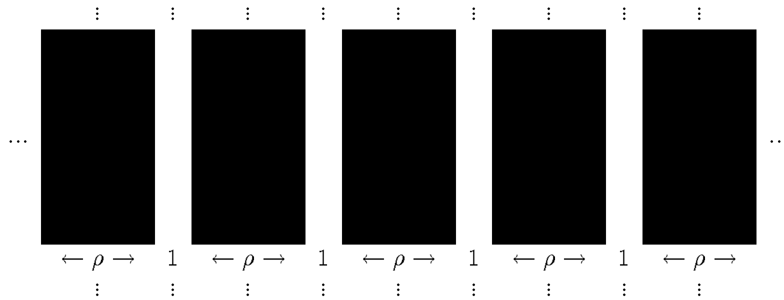
and this is the case that occurs when $\sigma=4\rho(\rho+1)$. Observe that for $M=\max\{\beta_2,\delta_2-1\}$,

$$\frac{4\rho(\rho+1)}{8\rho(\rho+1)-(\delta_2-1)}\leq\frac{4\rho(\rho+1)}{8\rho(\rho+1)-M}.$$

The latter quantity is the bound attained in Theorem 5.1.1.

Now let us show that the bound we obtain in Theorem 5.1.2 is attained by an entire set of LtL rules. To do this we need the following proposition.

Proposition 5.1.1. Let Λ be a configuration consisting of infinite strips of $1's$, each with width ρ, and separated by infinite strips of $0's$, each with width 1. That is,



Λ

Then Λ is a still life under any range ρ LtL rule such that $4\rho^2 - 1 \in [\delta_1, \delta_2]$ and $2\rho(2\rho + 1) \notin [\beta_1, \beta_2]$.

Proof. Suppose $\xi_0 = \Lambda$ and $\xi_0(x) = 1$. Then $|(x + \mathcal{N}) \cap \xi_0| = (2\rho + 1)^2 - 2(2\rho + 1) = 4\rho^2 - 1$ (we get equality because all of the occupied sites see exactly two strips of $0's$). Thus, by hypothesis, $\xi_1(x) = 1$. If $\xi_0(x) = 0$, then $|(x + \mathcal{N}) \cap \xi_0| = (2\rho + 1)^2 - (2\rho + 1) = 2\rho(2\rho + 1)$. Thus, by hypothesis, $\xi_1(x) = 0$. $\square$

The density of Λ is $\frac{\rho}{\rho+1}$ and it goes to 1 as $\rho \to \infty$. We point out that Λ will actually be fixed under any two-state CA rule, not restricted to the LtL family, provided a 1 survives when it sees $4\rho^2 - 1$ $1's$ and a 0 does not become a 1 when it sees $2\rho(2\rho + 1)$ $1's$.

One can construct many infinite still lifes similar to Λ, fixed under different LtL rules. This is done by varying the widths of the infinite strips of $0's$ and $1's$. We are interested in the one from Proposition 5.1.1 because it provides an example whose density is close to the bound obtained in Theorem 5.1.2. Let us show how its density compares to our bound. The bound is best if σ, and hence δ_2 is small, so let us do the case when $\delta_2 = 4\rho^2 - 1$. In that case, if $\rho \geq 2$ then $\sigma = 4\rho(\rho + 1) - 4$ (see Appendix I). Thus, by Theorem 5.1.2,

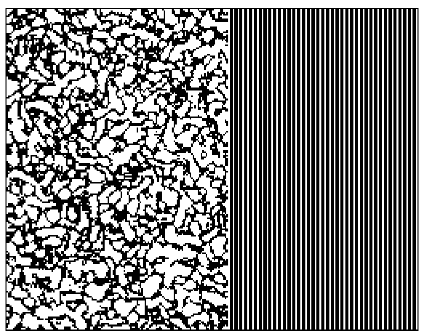
$$p \leq \frac{4\rho^2+4\rho-4}{4\rho^2+8\rho-2}.$$

The density of Λ is $\frac{\rho}{\rho+1}$, so if $\rho = 2$, then

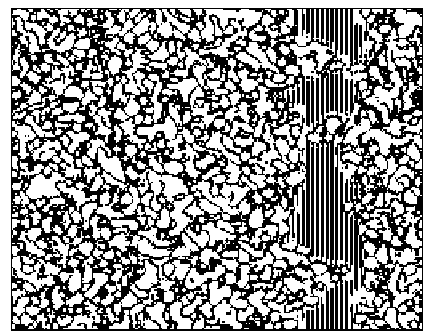
$$\frac{4\rho^2+4\rho-4}{4\rho^2+8\rho-2} = \frac{\rho}{\rho+1} = \frac{2}{3}.$$

In range 2 the bound is attained by all rules with $\delta_2 = 15$ and $20 \notin [\beta_1, \beta_2]$ since the Λ from Proposition 5.1.1 is an infinite still life for all such rules. Thus, if μ is the still life measure determined by Λ, then it has the largest possible density of any such measure.

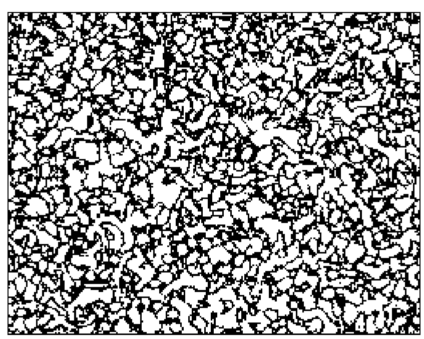
We point out, however, that starting a rule from a random initial state may often yield a density that is a lot smaller. For example, if we run the rule $(2, 10, 13, 6, 15)$ starting from product measure with density 0.5 for 100 time steps, the result is aperiodic dynamics with a density that is approximately 0.42. If we take that configuration and place a portion of Λ (depicted in the following pictures by black and white stripes) over part of it and then run the rule, the aperiodic dynamics beat up on the still life portion. The following figures show that, by time 50, the still life portion has been almost completely destroyed by the aperiodic dynamics.



time 0

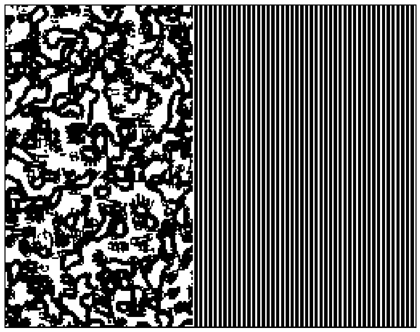


time 50

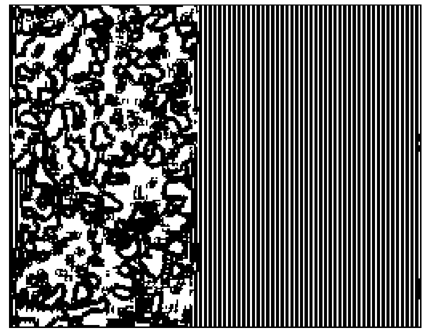


time 75

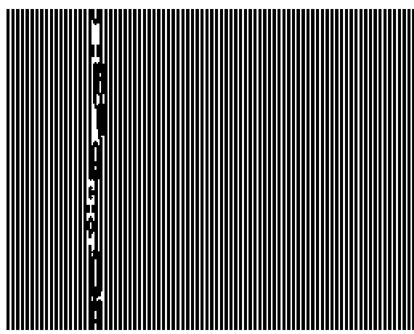
We see that, by time 75, all of Λ has been completely destroyed and the configuration looks as it did before Λ was inserted. Now let us vary the parameters, to the rule $(2, 5, 16, 2, 20)$, and place a portion of Λ on the aperiodic configuration generated by that rule after being run for 100 time steps on a random initial configuration with density 0.33. In this case, the portion of Λ grows, though very slowly, and eventually locks into periodicity where the two propagating edges meet (we used wrap-around boundary conditions). In the infinite system, it would fill in the lattice to yield an exact copy of Λ.



time 0

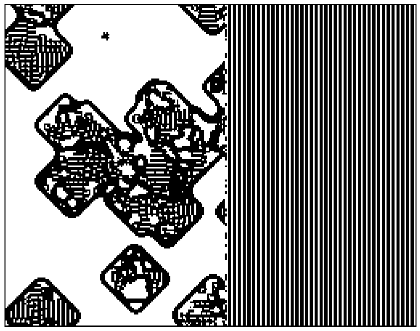


time 50

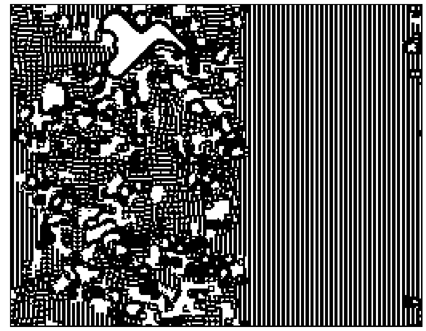


time 700

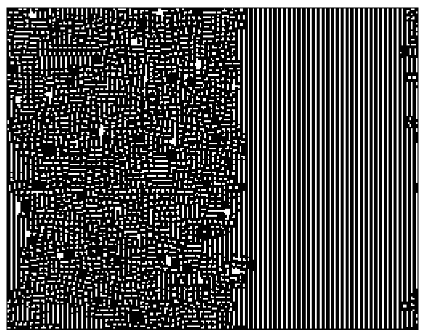
The third and final example we give is the rule $(2, 8, 18, 11, 22)$. Again we run the rule on an initial product measure, with density $\frac{1}{5}$ in this case. Then we insert a portion of Λ and use that as time 0. In this case, however, neither Λ, nor the other configuration "wins." Rather, the eventual state is locally periodic, with much of it fixed in a tile-like pattern that appears to be an approximation of Λ. If we had let this run indefinitely, from the random initial state, it also would have yielded a locally periodic limiting state (though probably not such a large chunk tiled by perfect stripes).



time 0



time 25

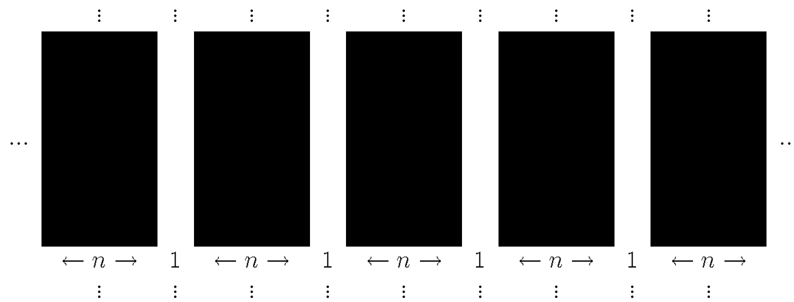


time 250

We illustrated the cases above to show that first, although the experimental density of a rule may be low, there may exist invariant sets with high densities that do not arise out of random initial states. Second, these examples show that, when started from random initial configurations, the limiting states for rules which admit invariant measures consisting of vertical stripes, vary dramatically. The last example we gave would seem to be the only "likely" candidate for such an invariant measure.

Let us describe three more infinite still lifes, one with a density that can be as close to 1 as we like, one with density $\frac{1}{2}$, and the third with a density as close to 0 as we like.

Proposition 5.1.2. Let Λ be a configuration consisting of infinite strips of $1's$, each with width n, $n \geq 2\rho$, and separated by infinite strips of $0's$, each with width 1. That is,



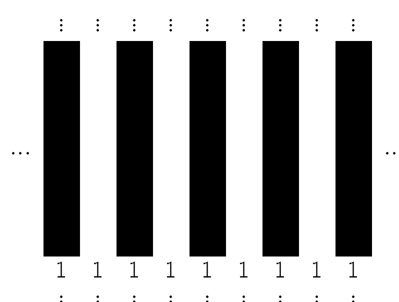
$\mathbf{\Lambda}$

Then Λ is a still life under any range ρ LtL rule such that both $2\rho(2\rho+1)$ and $(2\rho+1)^2 \in [\delta_1, \delta_2]$ and $2\rho(2\rho+1) \notin [\beta_1, \beta_2]$.

Proof. Suppose $\xi_0 = \Lambda$ and $\xi_0(x) = 1$. Then $|(x+\mathcal{N}) \cap \xi_0| = (2\rho+1)^2$ or $2\rho(2\rho+1)$ (we get equality because all of the occupied sites see either zero or one strips of $0's$, respectively). Thus, by hypothesis, $\xi_1(x) = 1$. If $\xi_0(x) = 0$, then $|(x+\mathcal{N}) \cap \xi_0| = (2\rho+1)^2 - (2\rho+1) = 2\rho(2\rho+1)$. Thus, by hypothesis, $\xi_1(x) = 0$. □

The density of the infinite still life from Proposition 5.1.2 is $\frac{n}{n+1}$, which goes to 1 as n goes to ∞. (Note that $n \geq 2\rho$ implies that n automatically goes to ∞ as the range does.) Since $\delta_2 = (2\rho+1)^2$, $\sigma = (2\rho+1)^2$ (see Appendix I), and Theorem 5.1.2 yields the bound, $p \leq 1$ (so our example agrees with the theorem).

Proposition 5.1.3. Let Λ be a configuration consisting of infinite strips of $1's$, each with width 1, and separated by infinite strips of $0's$, each with width 1. That is,



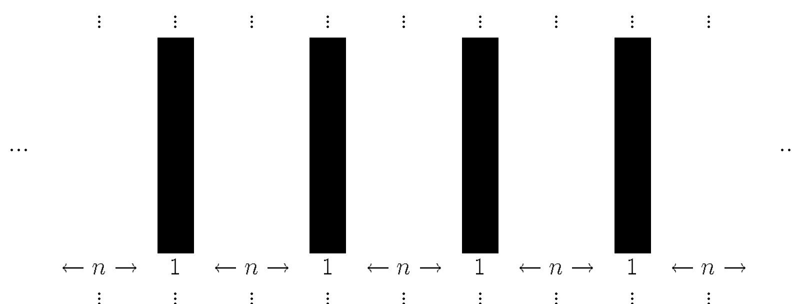
$\mathbf{\Lambda}$

Then Λ is a still life under any range ρ LtL rule such that if ρ is odd, then $\rho(2\rho+1) \in [\delta_1, \delta_2]$ and $(\rho+1)(2\rho+1) \notin [\beta_1, \beta_2]$, or, if ρ is even, then $(\rho+1)(2\rho+1) \in [\delta_1, \delta_2]$ and $\rho(2\rho+1) \notin [\beta_1, \beta_2]$.

Proof. Suppose $\xi_0 = \Lambda$ and $\xi_0(x) = 1$. If ρ is odd then $|(x + \mathcal{N}) \cap \xi_0| = \rho(2\rho + 1)$ (we get equality because all of the occupied sites see exactly ρ strips of $1's$). Thus, by hypothesis, $\xi_1(x) = 1$. If $\xi_0(x) = 0$, then $|(x + \mathcal{N}) \cap \xi_0| = (\rho + 1)(2\rho + 1)$ (we get equality because all of the $0's$ see exactly $\rho + 1$ strips of $1's$). Thus, by hypothesis, $\xi_1(x) = 0$. If ρ is even, the $1's$ see exactly $\rho + 1$ strips of $1's$ and the $0's$, exactly ρ strips of $1's$. □

For any range 1 rule with $\delta_2 = 3$ and $6 \notin [\beta_1, \beta_2]$ the Λ from Proposition 5.1.3 is an infinite still life. The density of Λ is $\frac{1}{2}$. Since $\delta_2 = 3$ Appendix I gives $\sigma = 6$. By Theorem 5.1.2 the density, p, of any still life measure for any range 1 rule with $\delta_2 = 3$ and $6 \notin [\beta_1, \beta_2]$ satisfies $p \leq \frac{1}{2}$. Thus, again we have a set of examples that attain the bound given in Theorem 5.1.2.

Proposition 5.1.4. Let Λ be a configuration consisting of infinite strips of $0's$, each with width n, $n \geq 2\rho$ and separated by infinite strips of $1's$, each with width 1. That is,



$\boldsymbol{\Lambda}$

Then Λ is a still life under any range ρ LtL rule such that $2\rho + 1 \in [\delta_1, \delta_2]$ and $2\rho + 1 \notin [\beta_1, \beta_2]$ (and $\beta_1 \neq 0$).

Proof. Suppose $\xi_0 = \Lambda$ and $\xi_0(x) = 1$. Then $|(x + \mathcal{N}) \cap \xi_0| = 2\rho + 1$ (we get equality because all of the occupied sites see exactly one strip of $1's$). Thus, by hypothesis, $\xi_1(x) = 1$. If $\xi_0(x) = 0$, then $|(x + \mathcal{N}) \cap \xi_0| = 2\rho + 1$ or 0. Thus, by hypothesis, $\xi_1(x) = 0$. □

The density of the infinite still life from proposition 3 is $\frac{1}{n+1}$, which goes to 0 as n goes to ∞. (Note that $n \geq 2\rho$ implies that n automatically goes to ∞ as the range does.)

Rules that admit finite still lifes also admit a large number of still life measures. This is part of the reason we call them still life measures -- using finite still lifes, we are able to construct a huge number of still life measures. To illustrate this point, let us do one such construction.

Let $\Lambda \subset \mathbb{Z}^2$ be a $\lambda_1 \times \lambda_2$ rectangle with periodic boundary conditions. Assume that Λ is painted in such a way that everything is fixed under ξ_t. That is,

$$\xi_t^{\Lambda \cap \xi_0} = \Lambda \cap \xi_0 \text{ for all times } t.$$

Tile $\mathbb{Z}^2$ with Λ, beginning by placing one of the vertices of Λ at the origin and forcing the rest of its elements to have coordinates that are greater than or equal to zero. That is,

$$\Lambda = \{x = (x_1, x_2) \in \mathbb{Z}^2 : 0 \leq x_i < \lambda_i,\ i = 1, 2, ..., d\}.$$

The remaining tiles are identically oriented, so that the sites in each tile see the equivalent of the assumed periodic boundary conditions (see the following figure). Then the tiling, which we denote by $\widetilde{\Lambda}$, is fixed under ξ_t. There are $\lambda_1\lambda_2$ distinct shifts of the tiling. Form the average of all the shifts,

$$\mu \equiv \frac{1}{\lambda_1 \lambda_2} \sum_{v \in \mathcal{R}} \theta^v(\widetilde{\Lambda}),$$

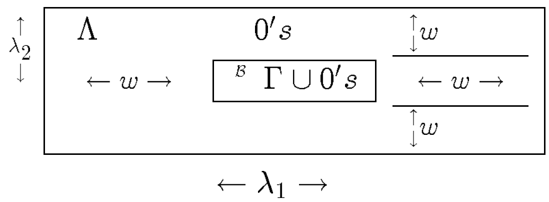
where $v = \alpha_1 e_1 + \alpha_2 e_2$, $(0 \leq \alpha_i < \lambda_i,\ i = 1, 2)$ is a vector in Λ, and θ^v is the shift operator which translates the entire tiling to the right $\alpha_1$units, and up α_2 units. Then, by construction, μ is translation invariant and is fixed under ξ_t in the sense that $P_\mu(\xi_1 = \xi_0) = 1$.

		$\vdots$		
	Λ	Λ	Λ	
$\ldots$	Λ	Λ	Λ	$\ldots$
	Λ	Λ	Λ	
		$\vdots$		

Tiling of $\mathbb{Z}^2$ by Λ.

Theorems 5.1.1 and 5.1.2 give upper bounds for various non-trivial invariant measures. All meaningful rules admit the trivial still life measure since any finite set consisting of all $0's$ is fixed for all such rules. The question thus arises: Can we obtain lower bounds for the non-trivial invariant measures? For starters, Proposition 5.1.4 shows that for any rule with $2\rho + 1 \in [\delta_1, \delta_2]$ and $2\rho + 1 \notin [\beta_1, \beta_2]$ (and $\beta_1 \neq 0$), there exists a still life measure with density that can be as small as we like, by taking the number of infinite strips of $0's$ in Λ to be as large as we like. We can also show that there are more rules for which we can construct still life measures with densities that are as small as we like. To illustrate this, let us show how one can construct a still life measure, with density as small as we like, from a finite still life.

Let Γ be a finite still life under ξ_t. Then $|\Gamma| = n < \infty$, $\xi_t^\Gamma = \Gamma$, and there exists a rectangle, $\Lambda \subset \mathbb{Z}^2$, with dimensions $\lambda_1 \times \lambda_2$ such that $\Gamma \subset \Lambda$. Choose Λ so that Γ fits inside in such a way that there is a band of $0's$ of width, $w \geq \rho/2$, surrounding the smallest rectangle, $\mathcal{B}$, that contains all of Γ. Place Γ inside Λ and fill in the remainder of Λ with $0's$ as follows:



If Λ has periodic boundary conditions and is painted as described above, then everything in it is fixed under ξ_t. Thus, we can use Λ to construct a still life measure μ.

Observe that, for each fixed pair (ξ_t, Γ), where Γ is a finite still life under ξ_t, there is a family of still life measures, each of which is determined by the size of the rectangle Λ, which is described above, and the placement of Γ inside Λ. We can make the densities of these measures as small as we like by increasing the width, w, of the band of $0's$ surrounding Γ in Λ. Hence, for a fixed Γ, a rectangle Λ, and the still life measure, μ, that they determine, simply increasing the dimensions of Λ yields another still life measure $\tilde{\mu}$ that has smaller density. Thus, there is no positive lower bound on the densities of still life measures.

The above discussion shows that invariant measures for rules which admit still lifes do not have lower bounds. Suppose we look at a set of rules which do not admit still lifes. Do their invariant measures have lower bounds? We claim that the answer is yes, if we add the condition that the set of rules admit neither periodic objects, nor bugs. How does one come up with even one such rule? It is necessary to check that for such a candidate rule, any seed started on a background of $0's$ either shrinks and eventually dies, or grows forever, covering $\mathbb{Z}^2$ with all $1's$ or some pattern, with density less than 1, of $1's$. We have discovered a couple of rules which appear through empirical data to have these properties. However, we are not yet convinced that the rules are indeed examples.

We conclude this chapter with the construction of a period two measure along with a theorem that gives an upper bound on the density of the measure.

Construction of a period 2 measure.

Let $\Lambda \subset \mathbb{Z}^2$ be a $\lambda_1 \times \lambda_2$ rectangle with periodic boundary conditions. Assume that Λ is painted in such a way that everything in it is period 2 under ξ_t. That is, every site changes state every time step. As we did in the construction of a still life measure, tile $\mathbb{Z}^2$ with Λ, assuming that each tile is identically oriented, so that the sites in each tile see the equivalent of the assumed periodic boundary conditions. Then every site in the tiling, which we denote by $\widetilde{\Lambda}$, flip flops every time step under ξ_t. There are $\lambda_1\lambda_2$ distinct shifts of the tiling. Form the average of all the shifts,

$$\mu \equiv \frac{1}{\lambda_1\lambda_2} \sum_{v\in\Lambda} \theta^v(\widetilde{\Lambda}),$$

where $v = \alpha_1 e_1 + \alpha_2 e_2$, $(0 \le \alpha_i < \lambda_i, i = 1, 2)$ is a vector in Λ, and θ^v is the shift operator which translates the entire tiling to the right $\alpha_1$units, and up α_2 units. Then, by construction, μ is translation invariant and it is period 2 under ξ_t in the sense that $P_\mu(\xi_1 = \xi_0) = 1$.

Theorem 5.1.3. Let $\Lambda \subset \mathbb{Z}^2$ be a $\lambda_1 \times \lambda_2$ rectangle with periodic boundary conditions. Assume that all sites in Λ are period 2 under ξ_t. Let μ be the period 2 measure determined by Λ. Let $p = \mu(\xi(x) = 1) = P_\mu(\xi_0(x) = 1)$ and $q = \mu(\xi(x) = 0) = 1 - p$. Then

$$p \le \frac{4\rho(\rho+1)-\beta_1}{8\rho(\rho+1)-\beta_1-\beta_2}, \ (8\rho(\rho+1) - \beta_1 - \beta_2 \neq 0).$$

Proof. Let $x, y \in \Lambda$. As in the proof of the previous theorem,

$$\sum_{y\in x+\mathcal{N}} P_\mu(\xi_0(y) = 1) = E(\sum_{y\in x+\mathcal{N}} 1_{\xi_0(y)=1} \cdot 1_{\xi_1(x)=1}) +$$

$$E(\sum_{y\in x+\mathcal{N}} 1_{\xi_0(y)=1} \cdot 1_{\xi_1(x)=0})] \le \beta_2 p + [4\rho(\rho+1) - \beta_1](1-p).$$

The first part of the inequality holds because $\xi_1(x) = 1$ implies that $\xi_0(x) = 0$. Thus,

$$\beta_1 \le |\xi_0 \cap (x + \mathcal{N})| \le \beta_2.$$

The second part of the inequality holds because $\xi_1(x) = 0$ implies that

$$\xi_0(x) = \xi_2(x) = 1.$$

Thus,

$$\beta_1 \le |\xi_1 \cap (x + \mathcal{N})| \le \beta_2$$

(in particular, x must see at least β_1 $1's$ at time 1). Since Λ is period 2, all sites flip every time step (so all of the $1's$ at time 0 become $0's$ at time 1). Thus,

$$4\rho(\rho+1) - \beta_2 \le |\xi_0 \cap (x + \mathcal{N})| \le 4\rho(\rho+1) - \beta_1.$$

(Otherwise, $|\xi_1 \cap (x + \mathcal{N})|$ will be strictly less than β_1.) We also have that

$$\sum_{y \in x+\mathcal{N}} P_\mu(\xi_0(y) = 1) = 4\rho(\rho + 1)p.$$

Combining these yields:

$$4\rho(\rho + 1)p \leq \beta_2 p + [4\rho(\rho + 1) - \beta_1](1 - p)$$

and hence the desired inequality. □