4.3 Birth or Death Only LtL Rules

Another simplifying feature of two state CA rules is for sites in one of the states to automatically switch to the other state at the next time step. There are two families of such LtL rules -- those for which $0's$ automatically become $1's$ and those for which $1's$ automatically become $0's$. Range ρ LtL rules of the first type have parameters

$$\beta_1 = 0,\ \beta_2 = 4\rho(\rho+1),\ \delta_1, \delta_2.$$

Range ρ rules for which $1's$ automatically become $0's$ have LtL parameters

$$\beta_1,\ \beta_2,\ \delta_1 = \delta_2 = 0.$$

Such rules are known as the two state *Greenberg-Hastings Model* (GHM).

$4.3 - A$. *Two-State Greenburg-Hastings Model*

In this section we will need to use the fact that a configuration on a set $A \subset \mathbb{Z}^2$ is an SPO iff for all x in A with $\xi_t(x) = 0$, $\beta_1 \leq |(x+N) \cap A \cap \xi_t| \leq \beta_2$, and for all x in A with $\xi_t(x) = 1$, either $|(x+N) \cap A \cap \xi_t| < \delta_1$ or $|(x+N) \cap A \cap \xi_t| > \delta_2$ for all times t.

The following theorem is the *two-state* version of a lemma in [FGG].

Theorem 4.3.1. If $\rho \geq 1$, $\beta_1 = 1$, $\beta_2 = 4\rho(\rho+1)$, and either $\delta_1 = \delta_2 = (2\rho+1)^2$ (CCA), or $\delta_1 = \delta_2 = 0$ (GHM), then starting from product measure with density p $(0 < p < 1)$ the rule is locally periodic of period 2 with probability one.

Proof. We need to show that

(i) each site is period 2 eventually in t and

(ii) $\inf_{t, x\neq y, a, b} P(\xi_t(x) = a, \xi_t(y) = b) > 0$.

To show (i), let C be the set of sites that have period 2 eventually in t. Observe that the configuration consisting of a 1 within range ρ of a 0 is an SPO and thus in C; hence, $C \neq \emptyset$. Suppose $x \notin C$, $y \in C$, and $||x-y|| \leq \rho$. Then $\exists$ a first time, τ, when $\xi_\tau(x) \neq \xi_\tau(y)$. But then, after time τ, the configuration consisting of x and y is an SPO. Hence, $x \in C$, which is a contradiction. Thus, every site belongs to C as claimed.

Now we show (ii). Given any time t, distinct sites x and y, and states a and b, the event in question will occur as long as there exists a pair of disjoint minimal SPOs through x and y in ξ_0. The probability of this is at least $\left(\frac{1}{pq}\right)^4 > 0$. $\square$

The proof of Theorem 4.3.1 showed that the existence of a minimal SPO was enough to enslave the other sites in the system into periodic cycles of length two. Are there other LtL rules which are uniformly locally periodic and admit minimal SPOs? If so, can we use the SPOs, as we did in the proof of Theorem 4.3.1, to prove local periodicity? Attempting to answer these questions led to the discovery of SPOs for various LtL rules; these are given in Proposition 4.3.1. It also led to the following two conjectures about uniform and nonuniform local periodicity. However, proving that all sites become enslaved to minimal SPOs becomes very tricky when β_1 is greater than one. We are convinced, nonetheless, that the following conjectures are true.

Proposition 4.3.1. The configuration consisting of exactly k $1's$ inside a box with side of length $\rho + 1$ $(k \leq \rho^2 + 2\rho)$ is an SPO with period 2 for the LtL rule with parameters

$$\beta_1 \leq \min(k, (\rho+1)^2 - k)), \beta_2 = 4\rho(\rho+1), \text{and}$$

(i) $1 \leq \delta_1 \leq \delta_2 \leq \min(k, (\rho+1)^2 - k)) - 1$ or

(ii) $\max((2\rho+1)^2 - k, 3\rho^2 + 2\rho + k) + 1 \leq \delta_1 \leq \delta_2 \leq (2\rho+1)^2$

Note that since there are only two colors, symmetry implies that the configuration consisting of exactly k $0's$ inside the box described above is the alternate phase of the SPO described above.

Proof. Let B be a box with side of length $(\rho+1)^2$ and containing exactly k $1's$. At time 0, each $x \in B$ has $k \leq |(x + \mathcal{N}) \cap \xi_0| \leq 3\rho^2 + 2\rho + k$. If $\xi_0(x) = 1$ then, since in (i) $k > \delta_2$ and in (ii) $\delta_1 > 3\rho^2 + 2\rho + k$, $\xi_1(x) = 0$. If, on the other hand, $\xi_0(x) = 0$ then since $\beta_1 \leq k$, $\xi_1(x) = 1$. Thus, there will be exactly $(\rho+1)^2 - k$ $1's$ in the box of side length $\rho + 1$ at time 1. Thus, at time 1, each $x \in B$ has $(\rho+1)^2 - k \leq |(x + \mathcal{N}) \cap \xi_0| \leq (2\rho+1)^2 - k$. If $\xi_1(x) = 1$ then since in (i) $(\rho+1)^2 - k > \delta_2$ and in (ii) $\delta_1 > (2\rho+1)^2 - k$, $\xi_2(x) = 0$. If, on the other hand, $\xi_1(x) = 0$ then since $\beta_1 \leq (\rho+1)^2 - k$, $\xi_1(x) = 1$ as desired. □

Conjecture 4.3.1. If $\rho \geq 1$, $k \leq \rho^2 + 2\rho$, $\beta_1 \leq \min(k, (\rho+1)^2 - k))$, $\beta_2 = 4\rho(\rho+1)$, and
(i) $1 \leq \delta_1 \leq \delta_2 \leq \min(k, (\rho+1)^2 - k)) - 1$ or
(ii) $\max((2\rho+1)^2 - k, 3\rho^2 + 2\rho + k) + 1 \leq \delta_1 \leq \delta_2 \leq (2\rho+1)^2$,
then starting from product measure with density p $(0 < p < 1)$ the rule is locally periodic of period 2, with probability one (in other words, it is uniformly locally periodic).

Observe that part (ii) of our conjecture contains some of the TVA rules, and those near the TVA (that is, when $\delta_2 = (2\rho + 1)^2$), and agrees with the conjectures about those rules, given in the section on symmetric rules.

Conjecture 4.3.2. If $\rho \geq 1$, $\beta_1 \leq 2\rho^2 + 2\rho$, $\beta_2 = 4\rho(\rho + 1)$, and
(i) $1 \leq \delta_1 \leq \delta_2 \leq \beta_1 - 1$ or
(ii) $(2\rho + 1)^2 - \beta_1 + 1 \leq \delta_1 \leq \delta_2 \leq (2\rho + 1)^2$,
then starting from product measure with density p $(0 < p < 1)$ the rule is locally periodic, with probability one (note that many sites may actually have period one).

Again part (ii) of our conjecture contains some of the TVA rules (that is, when $\delta_2 = (2\rho + 1)^2$), and those near the TVA agree with the earlier conjectures made about them.

4.4 Exactly θ rule

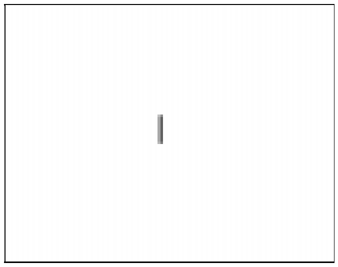
The *Exactly* θ set of rules was one of the first LtL cross sections we studied since keeping track of only one parameter greatly simplifies matters. A range ρ LtL rule in this set has parameters,

$$\beta_1 = \beta_2 = \delta_1 = \delta_2 = \theta,\ 0 \leq \theta \leq (2\rho + 1)^2.$$

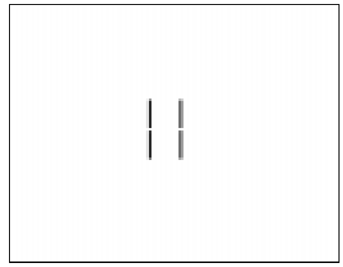
These rules are like the "derivative" of such *at least* θ rules as the TVA and TGM. We mean derivative in the sense that exactly rules map curves to curves, where as the at least rules map regions to regions. The quadratic rescaling of the at least rules is fairly coherent. That is, those rules can be thought of in terms of the proportions of sites required for birth and survival, with respect to the total number of neighbors,. However, the exactly rules rescale linearly, in terms of the range. For example, the range ρ exactly rule with $\theta = 2\rho$ has a similar limiting state in all ranges. In addition, in every range, the rule admits digital creatures, analogous to Life's gliders. Let us describe the dynamics exhibited by these rules.

- For $1 \leq \theta \leq \rho$, the dynamics are aperiodic. These rules also admit $2-$dimensional replicators. Let us give such a range 5 example. The initial state is a segment of $1's$ with length 11 and width 1.

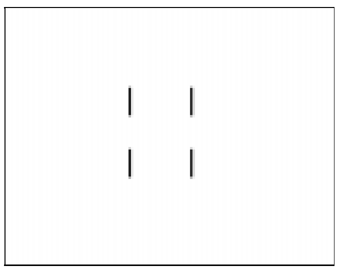
Two-dimensional replicator



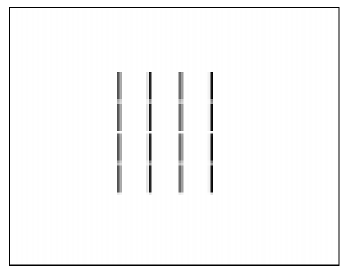
time 0



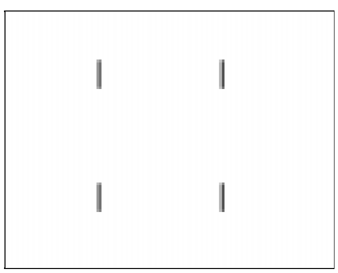
time 2



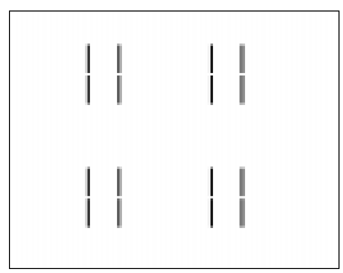
time 4



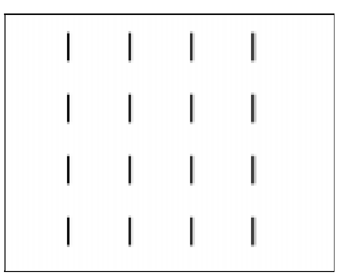
time 6



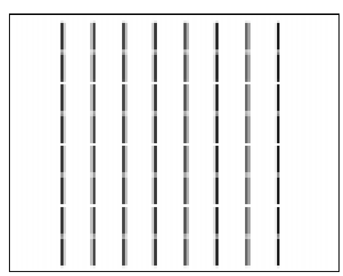
time 8



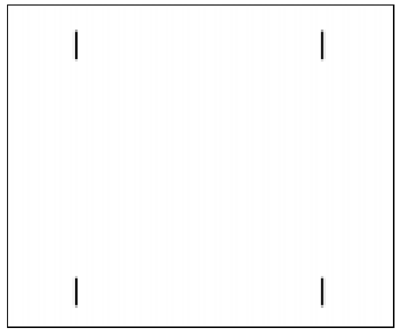
time 10



time 12



time 14



time 16

For more information on replicators, see Chapter 6. See Appendix II for a range ρ generalization of the above example.

- For $\theta = 2\rho - 1$, there exist bugs that are similar to Life's gliders.

- For $\rho \geq 2$, $\theta = 2\rho$ the dynamics are Life-like, meaning that eventually only bugs, still lifes, and periodic objects remain. In some cases, the objects eventually annihilate one another resulting in all $0's$. In range 2, there are so many bugs that the collisions among them result in aperiodic dynamics. The dynamics never seem to die out -- bugs move through the pockets of $0's$ that endlessly form amidst the aperiodicity. In all ranges, the rule admits bug makers similar to Life's glider guns (see Chapter 9).

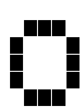
- There exist blinkers for $\theta = 2\rho + 1$. These are horizontal or vertical line segments composed of $2\rho + 1$ occupied sites. In range 1, these are Life's blinkers. Thus, although the LtL rule $(1, 3, 3, 3, 3)$ is not "Life," it admits one of Life's local configurations.

Life's blinker

times $2t$ $\leftrightarrow$ times $2t+1$

$(t = 0, 1, 2, ...)$

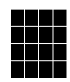
- For $\theta = 3\rho$, there exist finite still lifes. For example, in range 3, the still life is,



range 3 *still life*

In general, the still life lies along the edges of a square with side length $\rho + 2$, and has ρ occupied sites along each of the 4 sides, with the 4 corners being unoccupied. (Note that these are generalizations of Life's *tub*.)

- For $\theta = (\rho + 1)^2$ there exist finite still lifes. Again these generalize a finite Life configuration -- the *block*. In general, these are $(\rho + 1) \times (\rho + 1)$ squares filled with $1's$. We depict the range 2 version:



range 2 *block*

- If ρ is odd and $\theta = \rho(2\rho + 1)$, or if ρ is even and $\theta = (\rho + 1)(2\rho + 1)$ there exists an infinite still life with density $\frac{1}{2}$. In both cases it consists of infinite strips of $1's$, each with width 1, and separated by infinite strips of $0's$, each with width 1 (Proposition 5.1.3).

- For $\theta = 4\rho^2 - 1$ there exists an infinite still life with density $\frac{\rho}{\rho+1}$. It consists of infinite strips of $1's$, each with width ρ, and separated by infinite strips of $0's$, each with width 1 (see Proposition 5.1.1).

- For $\theta \geq 2\rho^2 + 3\rho + 2$ global death occurs from any initial configuration. (See Theorem 4.2.2.)

- **Conjecture 4.4.1.** For $\theta \geq 2\rho^2 + 2\rho + 1$ global death occurs from any initial configuration.

We point out that the $\theta = 3\rho$ and $\theta = (\rho + 1)^2$ still lifes described above generalize Life's local configurations in two distinct ways. The first takes what is essentially a curve, into curves in higher ranges. The second takes an object with positive area, to its higher dimensional analogues. This gives further evidence of our claim that Life configurations rescale both linearly and quadratically.