4.2 Monotone LtL Rules

Another simplifying feature for a two state CA rule, $\mathcal{T}$, to have is *monotonicity* or *attractiveness*. $\mathcal{T}$ is said to be *monotone* or *attractive* if it maps larger sets of $1's$ to larger sets of $1's$. That is, if A and B are sets of $1's$ and $A \subset B$ then $\mathcal{T}(A) \subset \mathcal{T}(B)$. In order for this to occur with a range ρ LtL rule, the upper thresholds for birth and survival must be their maximums. In other words, such LtL rules will have parameters

$$\beta_1,\ \beta_2 = |\mathcal{N}| - 1,\ \delta_1, \text{ and } \delta_2 = |\mathcal{N}|.$$

Thus, the TVA is a monotone CA rule. Another well known monotone LtL rule is the *discrete threshold growth model* (TGM). Let us discuss known results for the TGM that are relevant to our current discussion of LtL rules.

4.2 – *A. Discrete threshold growth model*

A CA from the *discrete threshold growth model* (TGM) has range ρ LtL parameters:

$$\beta_1 = \theta,\ \beta_2 = |\mathcal{N}| - 1,\ \delta_1 = 1, \text{ and } \delta_2 = |\mathcal{N}|,\ (\theta \geq 1).$$

Gravner and Griffeath have determined that each of the TGM rules generates one of three types of dynamics – *subcritical*, *critical*, or *supercritical*. They have also determined exactly which TGM rules generate each type of dynamics. Let us present their definitions and results from [GG1].

Let $A_\infty = \mathcal{T}^\infty(A_0) = \cup_{t=0}^{\infty} A_t$.

- Say that the dynamics are *supercritical* if there exists a finite A_0 such that A_t eventually occupies every site in $\mathbb{Z}^2$, i.e. $A_\infty = \mathbb{Z}^2$.
- Say that the dynamics are *critical* if $A_\infty \neq \mathbb{Z}^2$ for every finite A_0 but, for any A_0 with finite *complement*, $A_\infty = \mathbb{Z}^2$.
- Say that the dynamics are *subcritical* if there exists a non-empty finite set, H (a *hole*), so that the dynamics cannot fill H, even when started from the initial set $A_0 = H^c$.

For the following, we need: $\iota(\mathcal{N}) = \max\{|\mathcal{N} \cap l| : l \text{ a line through the origin}\}$.

Proposition 4.2.1. Threshold growth dynamics with $-\mathcal{N} = \mathcal{N}$ are:

- supercritical iff $\theta \leq \frac{1}{2}(|\mathcal{N}| - \iota(\mathcal{N}))$,

- subcritical iff $\theta > \frac{1}{2}(|\mathcal{N}| - 1)$.
  Moreover, in the critical case, for every $p > 0$, the dynamics started from product measure with density p fill the lattice a.s.; i.e. $A_\infty = \mathbb{Z}^2$ a.s.

In the case of range ρ box neighborhoods, $-\mathcal{N} = \mathcal{N}$ and $\iota(\mathcal{N}) = 2\rho + 1$ so for LtL rules with parameters

$$\beta_1 = \theta,\ \beta_2 = |\mathcal{N}| - 1,\ \delta_1 = 1, \text{ and } \delta_2 = |\mathcal{N}|,\ (\theta \geq 1),$$

the above says that the dynamics are:

- supercritical if $\theta \leq 2\rho^2 + \rho$,
- critical if $\theta \in [2\rho^2 + \rho + 1, 2\rho^2 + 2\rho]$
- subcritical if $\theta > 2\rho^2 + 2\rho$.

In particular, this says that if $\theta \leq 2\rho^2 + \rho$, then the limiting state is $\underline{1}$. In addition, if $\theta \in [2\rho^2 + \rho + 1, 2\rho^2 + 2\rho]$, then started from product measure with any $p > 0$, the limiting state is $\underline{1}$ a.s.

4.2 – *B. LtL rules near TGM -- global survival and global death*

This section contains theorems which prove that certain LtL rules near the set of monotone rules (and, in some cases, in the set of monotone rules) result in global death and global survival, starting from any random initial configuration. It is not a coincidence that these results apply to sets of rules that lie close to the boundaries of the parameter space (meaning that at least some of the parameters are at, or near, their extremes). It is near those extremes that we find the least nonlinear, and hence, most tractable rules.

The two theorems of this section specify LtL parameters that ensure global survival and global death, respectively. In order to prove them, we adapt to the LtL rules some of the formalism used in [GG1] to prove the results listed above for the TGM.

Recall that, if $\xi \subset \mathbb{Z}^2$ is a set of $1's$ (on a background of $0's$), we define

$$\mathcal{T}(\xi) = \{x \in \xi^c : \beta_1 \leq |(x + \mathcal{N}) \cap \xi| \leq \beta_2\} \cup \{x \in \xi : \delta_1 \leq |(x + \mathcal{N}) \cap \xi| \leq \delta_2\}.$$

Let $\widetilde{\xi} \subset \mathbb{R}^2$ be any subset of $1's$ (on a background of $0's$). Define

$$\widetilde{\mathcal{T}}(\widetilde{\xi}) = \{x \in \widetilde{\xi}^c : \beta_1 \leq |(x+\mathcal{N}) \cap \widetilde{\xi}| \leq \beta_2\} \cup \{x \in \widetilde{\xi} : \delta_1 \leq |(x+\mathcal{N}) \cap \widetilde{\xi}| \leq \delta_2\},$$

where $\mathcal{N}$ is still the range ρ box neighborhood of the origin, and $|\cdot|$ continues to be cardinality.

Then $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ are *conjugate*. That is, if $B \subset \mathbb{R}^2$ is any set of $1's$ (on a background of $0's$), then

$$\widetilde{\mathcal{T}}(B) \cap \mathbb{Z}^2 = \mathcal{T}(B \cap \mathbb{Z}^2).$$

Now we define the *speeds* of half-spaces, filled either with $1's$ or $0's$, assuming (in both cases) that their complements consist of any possible configurations of $1's$. Suppose u is a two-dimensional unit vector in the sphere $\mathcal{S}^1 \subset \mathbb{R}^2$. Denote the half-space $H_u^- = \{x \in \mathbb{R}^2 : \langle x, u\rangle \leq 0\}$ ($\langle \cdot\, , \cdot \rangle$is the Euclidean dot product).

Definition 4.2.1. Let $\widetilde{\xi}_0 \subset \mathbb{R}^2$ be the set of $1's$ at time 0.

(i) Define the *speed*, $w^1(u)$, of H_u^- by

$$w^1(u) = \max\{\lambda \in \mathbb{R} : H_u^- + \lambda u \subseteq \widetilde{\mathcal{T}}(\widetilde{\xi}_0)\ \} \text{ for every } \widetilde{\xi}_0 \text{ such that } H_u^- \subseteq \widetilde{\xi}_0.$$

The maximum may not exist, since all sites in H_u^- may become $0's$ next time, or all sites in the system may become $1's$. If all sites in H_u^- become $0's$ next time, define $w^1(u) \equiv -\infty$. If all sites in the system become $1's$, define $w^1(u) \equiv \infty$.

(ii) Define the *speed*, $w^0(u)$, of H_u^- by

$$w^0(u) = \max\{\lambda \in \mathbb{R} : (H_u^- + \lambda u)^c \supseteq \widetilde{\mathcal{T}}(\widetilde{\xi}_0)\} \text{ for every } \widetilde{\xi}_0 \text{ such that } (H_u^-)^c \supseteq \widetilde{\xi}_0.$$

Again the maximum may not exist, since all sites in H_u^- may become $1's$ next time, or all sites in the system may become $0's$. If all sites in H_u^- become $1's$ next time, define $w^0(u) \equiv -\infty$. If all sites in the system become $0's$, define $w^0(u) \equiv \infty$.

Lemma 4.2.1. For any u, the following hold, provided $\rho \geq 1$.

(i) $1 \leq |\mathcal{N} \cap \{x : \langle x, u\rangle = 0\}| \leq 2\rho + 1$.

(ii) $\rho(2\rho+1) \leq |\mathcal{N} \cap \{x : \langle x, u\rangle < 0\}| \leq 2\rho(\rho+1)$.

(iii) $2\rho^2 + 2\rho + 1 \leq |\mathcal{N} \cap \{x : \langle x, u\rangle \leq 0\}| \leq 2\rho^2 + 3\rho + 1$.

Proof. Let $E = \{|\mathcal{N} \cap l| : l \text{ is a line through the origin}\}$. For any u, since $\mathcal{N}$ is the range ρ box neighborhood, $1 \leq E \leq 2\rho + 1$. This proves (i). Combining the symmetry of $\mathcal{N}$ with (i) gives

$$\tfrac{1}{2}((2\rho+1)^2 - (2\rho+1)) \leq |\mathcal{N} \cap \{x : \langle x, u\rangle < 0\}| \leq \tfrac{1}{2}((2\rho+1)^2 - 1),$$

which is (ii). Again combining the symmetry of $\mathcal{N}$ and (i), we get

$$1 + \tfrac{1}{2}((2\rho+1)^2 - 1) \leq |\mathcal{N} \cap \{x : \langle x, u\rangle \leq 0\}| \text{ and}$$
$$|\mathcal{N} \cap \{x : \langle x, u\rangle \leq 0\}| \leq 2\rho + 1 + \tfrac{1}{2}((2\rho+1)^2 - (2\rho+1)),$$

which is (iii). □

Proposition 4.2.2. Assume $\rho \geq 1$.
(i) If $\delta_1 \leq 2\rho^2 + 2\rho + 1$, and $\delta_2 = (2\rho+1)^2$ then $w^1(u) \geq 0$ for every u.
(ii) If $\beta_1 > 2\rho(\rho+1)$ then $w^0(u) \geq 0$ for every u.

Proof. (i) If every site in H_u^- remains a 1 at the next time step, then $w^1(u) \geq 0$ for every u. (This holds since we are assuming that $H_u^- \subset \tilde{\xi}_0$.) Let $y \in \mathbb{R}^2$ be in a small neighborhood of the origin. Then for any u,

$$|(y + \mathcal{N}) \cap \{x : \langle x, u\rangle \leq 0\}| = |\mathcal{N} \cap \{x : \langle x, u\rangle \leq 0\}| \geq 2\rho^2 + 2\rho + 1.$$

The inequality holds by Lemma 4.2.1, part (iii). Thus, if $\delta_1 \leq 2\rho^2 + 2\rho + 1$, and $\delta_2 = (2\rho+1)^2$, then every site in H_u^- will remain a 1 next time.

(ii) If every site in H_u^- remains a 0 at the next time step, then $w^0(u) \geq 0$ for every u. (This holds since we are assuming that $(H_u^-)^c \supset \tilde{\xi}_0$.) Let $y \in \mathbb{R}^2$ be in a small neighborhood of the origin. Then for any u,

$$|(y + \mathcal{N}) \cap \{x : \langle x, u\rangle > 0\}| = |\mathcal{N} \cap \{x : \langle x, u\rangle > 0\}| \leq 2\rho(\rho+1).$$

The inequality holds by combining the symmetry of $\mathcal{N}$ with Lemma 4.2.1, part (ii). Thus, if $\beta_1 > 2\rho(\rho+1)$ then every site in H_u^- will remain a 0 next time. □

We say that LtL dynamics are *space-filling* if there exists a finite A_0 such that A_t eventually occupies every site in $\mathbb{Z}^2$, i.e. $A_\infty = \mathbb{Z}^2$. In particular, this implies that A_∞ exists. We say that the dynamics are *space-emptying* if there exists a co-finite set C_0, such that C_t eventually occupies no site in $\mathbb{Z}^2$, i.e. $C_\infty = \emptyset$. The first definition represents the case where sets of $1's$ do supercritical growth, and the second, where sets of $0's$ do supercritical growth (see [GG] and [GG1]). These are relevant to the LtL family because a large proportion of the rules result in global survival or global death.

Proposition 4.2.3.
(i) LtL dynamics are space-filling iff $w^1(u) > 0$ for every u.
(ii) LtL dynamics are space-emptying iff $w^0(u) > 0$ for every u.

Proof. (i) $w^1(u) > 0$ implies that there exists a bounded subset $A \subset \mathbb{R}^2$ so that $\widetilde{\mathcal{T}}^t(A) \uparrow \mathbb{R}^2$ (see Lemma 1 on p. 853 of [GG]). The converse is obvious since an edge speed of 0 constrains the spread of $1's$ to a corresponding half-space.

(ii) $w^0(u) > 0$ implies that there exists a co-finite subset $B \subset \mathbb{R}^2$ so that $\widetilde{\mathcal{T}}^t(B) \downarrow \emptyset$ (see Lemma 1 on p. 853 of [GG]). The converse is obvious since an edge speed of 0 constrains the spread of $0's$ to a corresponding half-space. □

The above shows that if one wants to prove that a large ball of $1's$ (or $0's$) does supercritical growth, it suffices to show that any half-space grows (or shrinks) linearly.

Theorem 4.2.1. If $\rho \geq 1$, $\beta_1 \leq \rho(2\rho+1)$, $\beta_2 = 4\rho(\rho+1)$, $\delta_1 \leq \rho(2\rho+1)+1$, and $\delta_2 = (2\rho+1)^2$, then starting from any random initial state, the limiting state is $\underline{1}$.

Proof. By Proposition 4.2.3, it suffices to show that $w^1(u) > 0$ for every u. Since $\delta_1 \leq \rho(2\rho+1)+1 \leq 2\rho^2+2\rho+1$, and $\delta_2 = (2\rho+1)^2$, Proposition 4.2.2 gives $w^1(u) \geq 0$ for every u. Thus, we need only show that $w^1(u) \neq 0$ for every u. Since we are working with $w^1(u)$, we assume $H_u^- \subset \widetilde{\xi}_0$. Let $y \in \mathbb{R}^2$ be in a small neighborhood of the origin. Then, for any u,

$$|(y+\mathcal{N}) \cap \{x : \langle x, u\rangle < 0\}| = |\mathcal{N} \cap \{x : \langle x, u\rangle < 0\}| \geq \rho(2\rho+1).$$

I.e. there are at least $\rho(2\rho+1)$ occupied sites in $y's$ neighborhood. Thus, if $y \in (H_u^-)^c$ was a 0, it will become a 1 if $\beta_1 \leq \rho(2\rho+1)$. If $y \in (H_u^-)^c$ was a 1, it will remain a 1 if $\delta_1 \leq \rho(2\rho+1)+1$. □

Theorem 4.2.2. If $\rho \geq 1$, $2\rho^2+3\rho+1 \leq \beta_1 \leq \beta_2 \leq 4\rho(\rho+1)$, and $2\rho^2+3\rho+1 < \delta_1 \leq \delta_2 \leq (2\rho+1)^2$, then starting from any random initial state, the limiting state is $\underline{0}$.

Proof. By Proposition 4.2.3, it suffices to show that $w^0(u) > 0$ for every u. Since $\beta_1 > \rho(2\rho+1)$, Proposition 4.2.2 gives $w^0(u) \geq 0$ for every u. Thus, we need only show that $w^0(u) \neq 0$ for every u. Since we are working with $w^0(u)$, we assume $(H_u^-)^c \supset \widetilde{\xi}_0$. Let $y \in \mathbb{R}^2$ be in a small neighborhood of the origin. Then, for any u,

$$|(y+\mathcal{N})\cap\{x:\langle x,u\rangle\geq 0\}|=|\mathcal{N}\cap\{x:\langle x,u\rangle\geq 0\}|\leq 2\rho^2+3\rho+1.$$

I.e. there are at most $2\rho^2+3\rho+1$ occupied sites in $y's$ neighborhood. Thus, if $y\in(H_u^-)$ was a 1, it will become a 0 if $\delta_1>2\rho^2+3\rho+1$. If $y\in(H_u^-)^c$ was a 0, it will remain a 0 if $\beta_1\geq 2\rho^2+3\rho+1$ (knowing y is a 0, allows us to subtract one site from the count). ☐

Corollary 4.2.1. If $\rho\geq 1$, $2\rho^2+3\rho+1\leq\beta_1\leq\beta_2\leq 4\rho(\rho+1)$, and $\delta_1=\delta_2=0$, then starting from any random initial state, the limiting state is $\underline{0}$.

Proof. This follows from Theorem 4.2.2 since in this case $1\rightarrow 0$ automatically so survival is even more difficult. ☐

Conjecture 4.2.1. If $\rho\geq 1$, $\beta_1\in[2\rho^2+\rho+1,2\rho^2+2\rho]$, $\beta_2=4\rho(\rho+1)$, $\delta_1\in[2,2\rho^2+2\rho+1]$, and $\delta_2=(2\rho+1)^2$, then for $p\in(p(\delta_1),1]$ starting from product measure with density p, the limiting state is $\underline{1}$ a.s. ($p(\delta_1)\geq 0$ depends on δ_1 -- if, as in threshold growth $\delta_1=1$, then $p(\delta_1)=0$).

Conjecture 4.2.1 is like Proposition 4.2.1 (taken from [GG1]), but in our case, survival is not guaranteed. (Proposition 4.2.1 is the $\delta_1=1$, $p(\delta_1)=0$ case.) Thus, we cannot use the bootstrap methods (see [AL]) that are crucial to the proof of Proposition 4.2.1. The bootstrap methods are used to show that a large set of $1's$ can grow, provided it gets some "help" from the $1's$ in the initial product measure surrounding it. The argument thus relies on the survival of sparse sets of $1's$ from the random initial state. However, by choosing the $p(\delta_1)$ of our conjecture carefully, we may still be able to adapt some of those ideas.

The following conjecture is similar to the previous one. However, it is about sets of $0's$ (rather than $1's$) bootstrapping their way to global death (rather than global survival).

Conjecture 4.2.2. If $\rho\geq 1$, $2\rho^2+2\rho+1\leq\beta_1\leq\beta_2\leq 4\rho(\rho+1)$, and $2\rho^2+2\rho+2\leq\delta_1\leq\delta_2\leq(2\rho+1)^2$, then for $p\in[0,p(\beta_1))$ starting from product measure with density p, the limiting state is $\underline{0}$ a.s. ($p(\beta_1)\leq 1$ depends on β_1).

What proportion of all LtL rules have we shown result in global survival? What about global death? Recall that, for a fixed range ρ, there are $(2k^2+3k+1)^2$ $(k=2\rho(\rho+1))$ possible LtL rules. We have shown that for all rules such that $\beta_1\leq\rho(2\rho+1)$,

$\beta_2 = 4\rho(\rho+1)$, $\delta_1 \leq \rho(2\rho+1)+1$, and $\delta_2 = (2\rho+1)^2$, the limiting state is $\underline{1}$. There are $(\rho(2\rho+1)+1)(\rho(2\rho+1)+2)$ such rules. Thus, such rules represent $\frac{(\rho(2\rho+1)+1)^2}{(2k^2+3k+1)^2}$ $(k = \frac{1}{2}(2\rho+1)^2)$ of the total.

$$\lim_{\rho\to\infty} \frac{(\rho(2\rho+1)+1)^2}{(2k^2+3k+1)^2} = 0.$$

In the case of global death, we have made more progress. We have shown that for all rules such that $2\rho^2 + 3\rho + 1 \leq \beta_1 \leq \beta_2 \leq 4\rho(\rho+1)$, and $2\rho^2 + 3\rho + 1 < \delta_1 \leq \delta_2 \leq (2\rho+1)^2$, the limiting state is $\underline{0}$. There are $(2l^2+3l+1)^2$ such rules, where $l = \rho^2 + \frac{1}{2}\rho - \frac{1}{2}$. Thus, such rules represent $\frac{(2l^2+3l+1)^2}{(2k^2+3k+1)^2}$ $(l = \rho^2 + \frac{1}{2}\rho - \frac{1}{2}$, $k = 2\rho(\rho+1))$ of the total. In this case,

$$\lim_{\rho\to\infty} \frac{(2l^2+3l+1)^2}{(2k^2+3k+1)^2} = \frac{1}{16}.$$

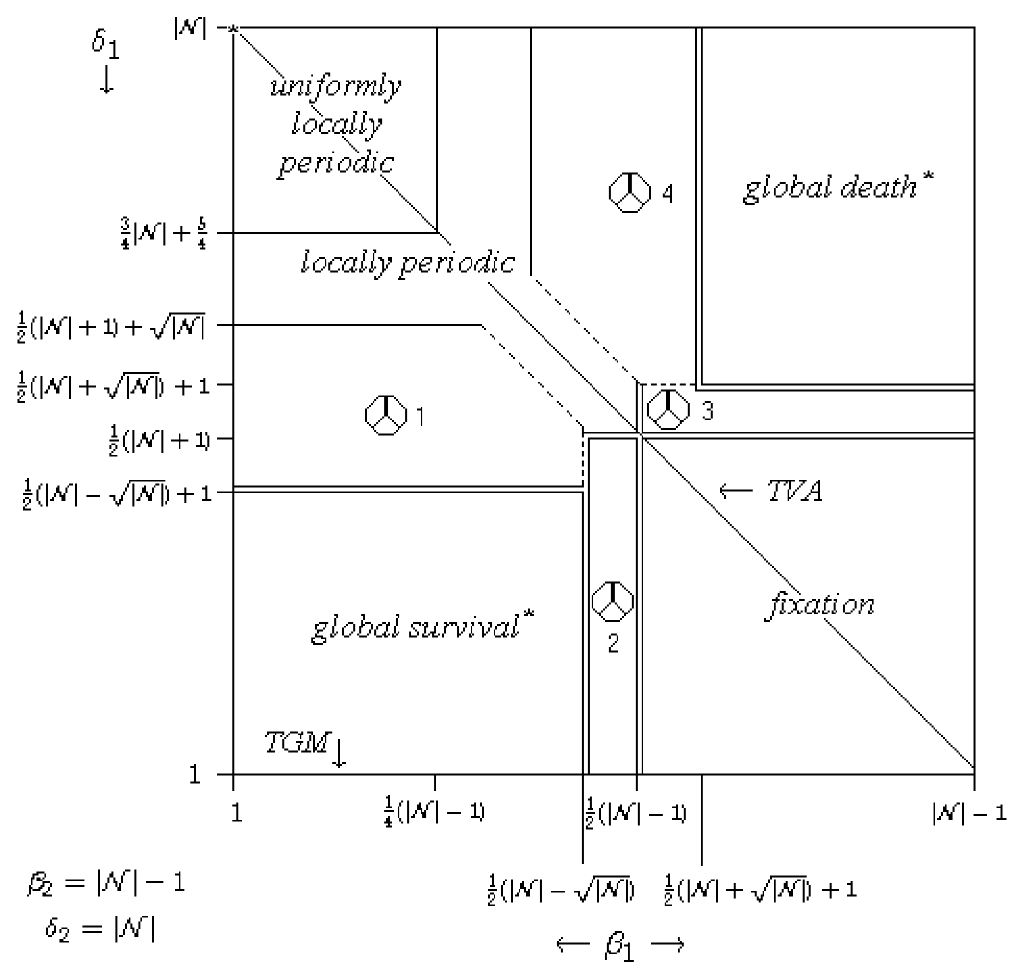
We also showed that an additional $2\rho^2 + \rho$ rules have limiting state $\underline{0}$. However, since this is quadratic in ρ, it does not change the above proportion which is our interest.

The limiting state $\underline{1}$ will be attained only if $\delta_2 = (2\rho+1)^2$. Thus, the set of LtL rules with this limiting state is contained in a three-dimensional cross section of LtL parameter space. Hence, it makes sense that the proportion of such rules goes to 0 as the range gets large. On the other hand, $\underline{0}$ does not have such a requirement, and we have shown that the set of rules with this limiting state live in a four-dimensional subspace. The moral? The rules are Larger than Life, but global survival is difficult. Perhaps a corollary to the above is: Death is Larger than Life.

How have the results obtained in this and Section 4.1 helped us better understand the geometry of LtL space in terms of the ergodic classifications of the rules? To answer this question let us depict the LtL cross section of monotone rules in terms of the ergodic classification at each point. The following diagram indicates the TVA set of rules with a diagonal line that cuts through the picture. The TGM rules are also indicated with a line at the bottom of the picture.

We indicate with a "*" the regions for which we have proved global death and global survival. There is also a "*" in the upper left corner of the diagram because we are going to prove in the next section (see Theorem 4.3.1) that the rule with parameters $\beta_1 = 1$, $\beta_2 = |\mathcal{N}| - 1$, and $\delta_1 = \delta_2 = |\mathcal{N}|$ is uniformly locally periodic. We indicate also the regions we believe to have locally periodic or fixed limiting states.

For the regions closer to the middle of the diagram, the ergodic classification is more complicated. In those cases, we use the warning icon to indicate the uncertainties. We believe a large portion of the rules in region one result in global survival. Gravner and Griffeath have shown that the limiting state for the rules in the intersection of region two and the TGM line result in global survival when started from product measure of any positive density. In Conjecture 4.2.1 we state that this is also the case for the remaining rules in region two, but only if the initial density is large enough. If the density is not large enough, we believe the limiting state is fixation. It may turn out that, in any case, the limiting state is fixation (see the discussion after Conjecture 4.2.1). In Conjecture 4.2.2 we state that the rules in region three result in global death provided the initial product measure has a small enough density. Otherwise we believe those rules fixate or are locally periodic. We believe that some of the rules in region four result in global death while others are locally periodic. The dashed lines indicate the regions for which the boundaries are especially murky.



Ergodic classification of monotone LtL rules (indicates that the limiting behavior has been proved).*