4 LtL Rules With Simplifying Features

In this chapter, we describe regions of LtL space where the rules have some sort of regularity property. In several cases, some of the LtL rules are well known. In those cases, we discuss the known rules, first interpreting them in terms of LtL parameters and then stating known results as well as unproved conjectures. In some cases, we perturb the results to a larger LtL subspace not contained in the known subspace. In other cases, we establish results and make conjectures about nearby LtL rules.

4.1 Symmetric LtL Rules

A simplifying feature for a two state CA rule to have is *symmetry* in the states. This means that the conditions for switching from each state to the other are the same. In the case of LtL rules, this means that the conditions for birth are the same as the conditions for death. Since LtL rules are defined by connected intervals for which birth and survival take place, it is easy to check that there are only two sets of symmetric LtL rules. These are the sets of rules for which both the birth and survival intervals are at one extreme or the other. That is, these are the range ρ box LtL rules with parameters that satisfy

(i) for $\tau \geq 1$,

$\beta_1 = \tau$, $\beta_2 = |\mathcal{N}| - 1$, $\delta_1 = |\mathcal{N}| - (\tau - 1)$, and $\delta_2 = |\mathcal{N}|$ or

(ii) for $\tau \geq 0$,

$\beta_1 = 0,\ \beta_2 = \tau,\ \delta_1 = 1,$ and $\delta_2 = |\mathcal{N}| - (\tau + 1)$.

(In this section it is convenient to use the notation $|\mathcal{N}| = (2\rho + 1)^2$.)

By (i), a site switches state if there are *at least* τ sites of the other state in its neighborhood. On the other hand, by (ii), a site changes state if it sees *at most* τ sites of the other state in its neighborhood. In that sense, these rules are complementary. (i) is called the *threshold voter automaton* (TVA) and (ii) is the *anti-voter automaton* (AVA). Let us discuss known results for the TVA.

4.1 – *A. Threshold voter automaton*

The TVA is typically defined as follows:

The *threshold voter automaton* is a deterministic discrete time process in which, at each time t, the voter at x examines the opinions of her neighbors in $x+\mathcal{N}$, and changes her state iff at least τ neighbors have the opposite opinion. (See [DS].)

In [DS] they use neighborhoods $\mathcal{N}=\{y: ||y||_p \leq \rho\}$, $p \in [1,\infty]$, where $||y||_p = (|y_1|+|y_2|+ ... +|y_d|)^{1/p}$, $p \in [1,\infty)$, $||y||_\infty = \sup_i |y_i|$. It is in this setting that they make the following conjectures and state the theorem. We are interested in the two-dimensional case $(d=2)$ with range ρ box neighborhoods $(p=\infty)$. As such, the following theorem and conjectures apply to range ρ LtL rule with parameters

$$\beta_1 = \tau\,,\ \beta_2 = |\mathcal{N}|-1,\ \delta_1 = |\mathcal{N}|-(\tau-1),\ \text{and}\ \delta_2 = |\mathcal{N}|\ \ (\tau \geq 1).$$

As may be clear from the name, in the TVA, switching states is thought of as switching political parties. (For what may be more appropriate to this election year, see [Gri] for a discussion of the multi-party case.) Based on a comparison with results about the one-dimensional TVA given in [FG], Durrett and Steif make the following "natural" conjecture (see [DS]):

Conjecture 4.1.1. The system:

(i) is uniformly locally periodic iff $\tau \leq \frac{|\mathcal{N}|-1}{4}$;

(ii) is locally periodic iff $\tau \leq \frac{|\mathcal{N}|-1}{2}$;

(iii) fixates iff $\tau \geq (\frac{|\mathcal{N}|-1}{2})+1$.

They have made a little progress on (iii) in the following theorem.

Theorem 4.1.1. Let B_k be the event that all sites x with $||x||_2 \leq k$ never change. Suppose $\tau = \theta|\mathcal{N}|$ with $\theta > \frac{3}{4}$. If we start from product measure with density $\frac{1}{2}$ then for all k, $P(B_k) \to 1$ as $\rho \to \infty$.

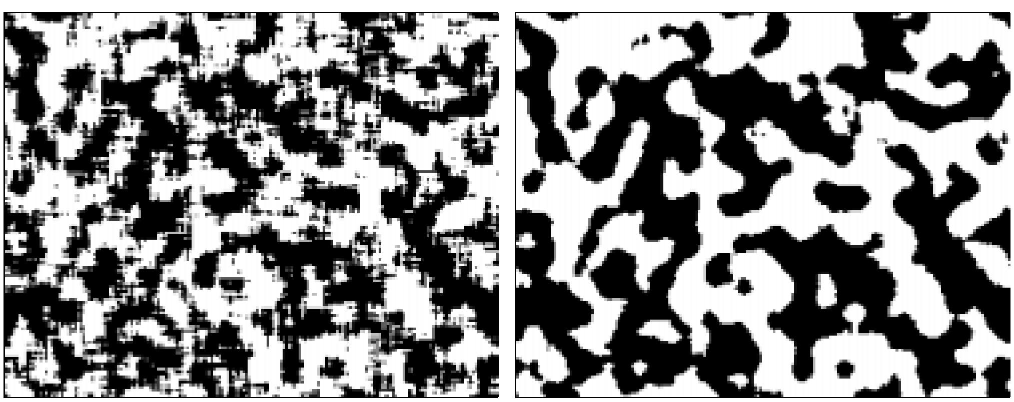
Theorem 4.1.1 is actually about sites that never change, and requires that the range go to ∞. They make a similar conjecture about fixation:

Conjecture 4.1.2. Let A_k be the event that all sites x with $||x||_2 \leq k$ fixate in the same state. Suppose $\tau = \theta|\mathcal{N}|$ with $\theta \in (\frac{1}{2}, \frac{3}{4})$. If we start from product measure with density $\frac{1}{2}$ then for all k, $P(A_k) \to 1$ as $\rho \to \infty$.

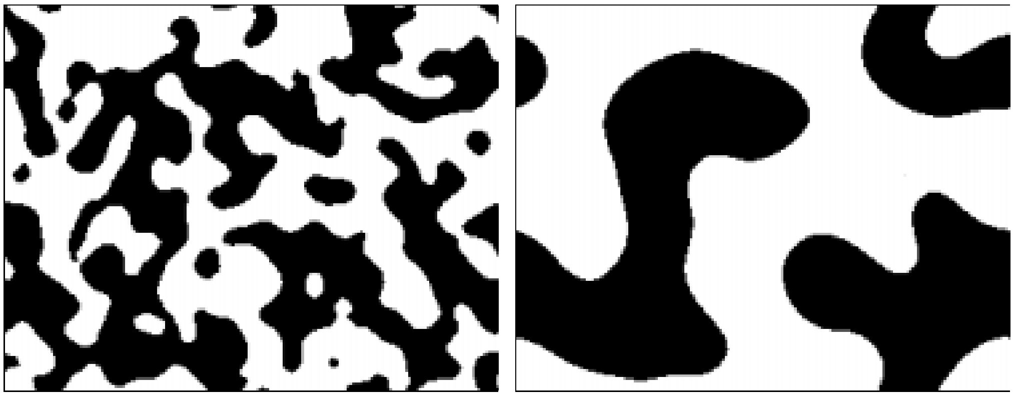
Note that the famous Majority Vote rule (where the voter changes her opinion iff she sees a *majority* of the opposite opinion in her neighborhood) is in the set of rules about which this conjecture applies. It is generally believed that the conjecture is true. However, due to the monkeys-at-the-typewriter principle, all sorts of proof-destroying configurations lurk and cause what may be paranoid delusions.

Some progress has been made, however, on understanding the dynamics of the Majority Vote rule. Let us illustrate times 1, 2, 3, and 40 of the range five version of the rule starting from a random initial state with equal densities of the two opinions (black represents the republicans and white the democrats).

Range 5 Majority Vote -- LtL rule $(5, 61, 120, 61, 121)$
(started from product measure with density $p = 0.5$).



time 1 time 2



time 3 time 40

As we see above, after just one time step, there is massive self-organization. After another update, there is a dramatic smoothing of the edges between the colors. At time three, there is a two-color tessellation of the array, and within the resolution of a range 5 discretization, it suggests that the boundaries are continuously differentiable. At times four and after, there is convexification, and erosion of bounded regions, according to dynamics that approximate a continuous flow known as *motion by mean curvature*. (For a better approximation to the continuous flow one must use a larger range.) By depicting time 40, we illustrate how this process continues until the boundaries achieve uniformly small curvature, at which point the system fixates. We chose to stop the evolution prior to fixation. Can the reader guess who wins in this case? Will it be the democrats? The republicans? Or will there be a stand-off between the two? We leave this as a little puzzle. In the style of Riemann, these CA approximations, suitably scaled, converge to Euclidean dynamics that cluster indefinitely. Results along these lines, as well as progress on the fixation of Majority Vote, are currently being formulated by Gravner and Griffeath.

Before returning to the current discussion, we make a few comments about the continuous time version of the TVA, known as the *threshold voter model.* It is defined in [DS] as follows:

There is an independent rate one Poisson process $\{T_n^x, n \geq 1\}$ for each lattice point $x \in \mathbb{Z}^d$. At time T_n^x, the voter at x examines the points in her neighborhood $\{y : y - x \in \mathcal{N}\}$. If at least τ neighbors have the opposite opinion then the opinion at x changes, otherwise it stays the same.

The following conjecture about this model is made in [DS]:

Conjecture 4.1.3. If $\mathcal{N} = \{y : ||y||_p \le \rho\}$ and $\tau = \theta|\mathcal{N}|$ with $\theta \in (\frac{1}{4}, \frac{1}{2})$, then for large ρ the system clusters starting from product measure with density $\frac{1}{2}$.

By adding randomness to the updating of the TVA, there is hope for clustering. On the other hand, there do not seem to be any LtL rules that cluster. Indeed, for a rule to cluster, it is necessary that large sets of $0's$ grow, as do large sets of $1's$. This requires a rule that is symmetric in $0's$ and $1's$. Thus, if there were an LtL rule that clustered, it would be a TVA or an AVA. Of those rules, Majority Vote seems to be the most likely candidate for clustering. As we have discussed, Majority Vote is known (empirically) to fixate, so we are out of luck. However, we still include the discussion because of the interesting paradox it poses: If a two state rule clusters, then regardless of which state a large connected component of one state is in, it will grow, but if it encounters a larger set of the other state, it will get devoured. The resolution is that although a large nonconvex set has the capability to grow, there is always something bigger of the other state out there that will beat it in a competition.

4.1 – B. *Anti-voter automaton*

All of the following statements are about the range ρ LtL rule with parameters

$$\beta_1 = 0,\ \beta_2 = \tau,\ \delta_1 = 1,\ \text{and } \delta_2 = |\mathcal{N}| - (\tau + 1)\ \ (\tau \ge 0).$$

For this rule, we make the following conjecture that is based on empirical data as well as comparison with the TVA. It is the AVA analogue to Conjecture 4.1.1.

Conjecture 4.1.4. Starting from any initial configuration, the system:

(i) is uniformly locally periodic iff $\tau \ge \frac{|\mathcal{N}|-1}{2}$;

(ii) is locally periodic iff $\tau \ge (\frac{|\mathcal{N}|-1}{4}) + 1$;

(iii) fixates iff $\tau \le \frac{|\mathcal{N}|-1}{4}$.

We also make the following conjectures about the AVA.

Conjecture 4.1.5. Let A_k be the event that for each x with $||x||_2 \leq k$, $\xi_t(x)$ eventually has period 2 in t. Suppose $\tau \geq \frac{|\mathcal{N}|-1}{2}$. If we start from product measure with density $\frac{1}{2}$ then for all k, $P(A_k) \to 1$ as $\rho \to \infty$.

Conjecture 4.1.6. Let C_k be the event that all sites x with $||x||_2 \leq k$, $\xi_t(x)$ fixate in the same state. Suppose $\tau = \theta(|\mathcal{N}| - 1)$ with $\theta \in [0, \frac{1}{4}]$. If we start from product measure with density $\frac{1}{2}$ then for all k, $P(C_k) \to 1$ as $\rho \to \infty$.

4.1 – *C. LtL Rules near the TVA or AVA -- fixation and local periodicity*

Now we extend the conjectures and theorem above to near-by LtL rules.

Conjecture 4.1.7. Suppose the range ρ LtL rule has parameters $\beta_2 = |\mathcal{N}| - 1$ and $\delta_2 = |\mathcal{N}|$. Then starting from any random initial configuration, the system

(i) is uniformly locally periodic if $\beta_1 \leq \frac{|\mathcal{N}|-1}{4}$, and $\delta_1 \geq \frac{3}{4}(|\mathcal{N}| - 1) + 2$;

(ii) is locally periodic if $\beta_1 \leq \frac{|\mathcal{N}|-1}{2} - \sqrt{|\mathcal{N}|}$, and $\delta_1 \geq (\frac{|\mathcal{N}|+1}{2}) + \sqrt{|\mathcal{N}|}$;

(iii) fixates if $\beta_1 \geq (\frac{|\mathcal{N}|-1}{2}) + 1$, and $\delta_1 \leq \frac{|\mathcal{N}|+1}{2}$.

Conjecture 4.1.8. Suppose the range ρ LtL rule has parameters $\beta_1 = 0$ and $\delta_1 = 1$. Then starting from any random initial configuration, the system

(i) is uniformly locally periodic if $\beta_2 \geq \frac{|\mathcal{N}|-1}{2}$, and $\delta_2 \leq \frac{|\mathcal{N}|-1}{2}$;

(ii) is locally periodic if $\beta_2 \geq (\frac{|\mathcal{N}|-1}{4}) + 1$, and $\delta_2 \leq \frac{3}{4}(|\mathcal{N}| - 1) - 1$;

(iii) fixates if $\beta_2 \leq \frac{|\mathcal{N}|-1}{4}$, and $\delta_2 \geq \frac{3}{4}(|\mathcal{N}| - 1)$.

We are not yet convinced that the previous two conjectures should contain the "only if" statements, as did those made about the TVA and AVA.

The following theorem is an extension of Theorem 4.1.1 to LtL rules near the TVA and a generalization to initial states with arbitrary density p $(0 < p < 1)$.

Theorem 4.1.2. Let B_k be the event that all sites x with $||x||_2 \leq k$ never change. Suppose the system starts from product measure with density p and that

$$\beta_2 = |\mathcal{N}| - 1 \text{ and } \delta_2 = |\mathcal{N}|.$$

If $p \geq \frac{1}{2}$, $\beta_1 > \frac{(1+p)}{2}|\mathcal{N}|$, and $\delta_1 < \frac{1}{4}|\mathcal{N}|$, or if $p \leq \frac{1}{2}$, $\beta_1 > \frac{3}{4}|\mathcal{N}|$, and $\delta_1 < \frac{p}{2}|\mathcal{N}|$ then for all k, $P(B_k) \to 1$ as $\rho \to \infty$.

To do the proof, we will need the following Lemma from [DS].

Lemma 4.1.1. Suppose $b < \frac{1}{2}$. There are constants R_0 and ρ_0 so that if $r \geq R_0$ and $\rho \geq \rho_0$, then each site x in $B_2(0, r\rho)$ has $|(x + \mathcal{N})| \cap B_2(0, r\rho)| \geq b|\mathcal{N}|$, where $B_2(0, r\rho) = \{x : ||x||_2 \leq r\rho\}$.

Lemma 4.1.1 essentially says that a large ball is locally flat. This means that sites inside the ball and near the boundaries see almost as many sites inside as they would have seen had they been inside a half-space.

Proof of Theorem 4.1.2. Consider a site $x \in B_2(0, r\rho)$ that is in state 0. Pick b and c so that

$$2(1 - \tfrac{\beta_1}{|\mathcal{N}|}) < 2c < b < \tfrac{1}{2}.$$

We also need the following large deviations result from Chapter 1 of [Dur1]:

> Let S_n be the sum of n i.i.d. random variables, X_1, X_2, ..., X_n, that take the value 1 with probability p and 0 with probability $1 - p$. Then for $c < p$,
>
> $$\lim_{n \to \infty} \frac{1}{n} \log P(S_n < cn) = -\gamma < 0,$$
>
> $$P(S_n < cn) \leq e^{-\gamma n} \text{ for all } n.$$

Observe that the above counts the number of i.i.d. random variables X_1, X_2, ..., X_n that are in state 1. It can also be used to count the number random variables that are in state 0 by replacing p with $1 - p$ and letting $S_n = \sum_{i=1}^{n}(1 - X_i)$.

Case 1a : $p \geq \frac{1}{2}$, $\beta_1 > \frac{(1+p)}{2}|\mathcal{N}|$, and $\delta_1 < \frac{1}{4}|\mathcal{N}|$. Assume $c < \frac{1-p}{2}$. (Note that the choice of c imposes the condition $2\left(1 - \frac{\beta_1}{|\mathcal{N}|}\right) < 1 - p$, which holds since we are assuming $\beta_1 > \frac{(1+p)}{2}|\mathcal{N}|$.)

Since $b < \frac{1}{2}$, the large deviations result stated above gives

$$\lim_{n\to\infty} \frac{1}{n} \log P(S_{bn} < cn) = -\gamma < 0.$$

(In the spirit of Durrett, γ is an unimportant constant that changes from line to line.)

Combining this with Lemma 4.1.1 gives that if $r \geq R_0$, then

P(all sites in $B_2(0, r\rho)$ have at least $c|\mathcal{N}|$ neighbors that are 0's) $\rightarrow 1$ as $\rho \rightarrow \infty$.
Thus, x will have at most $(1-c)|\mathcal{N}|$ 1's in its neighborhood. By the choice of c, $(1-c)|\mathcal{N}| < \beta_1$. Thus, x will not change state at the next time step. It follows that no 0 will become a 1.

Case 2a : If $p \leq \frac{1}{2}$, $\beta_1 > \frac{3}{4}|\mathcal{N}|$, and $\delta_1 < \frac{p}{2}|\mathcal{N}|$. Assume $c < \frac{p}{2}$. (Note that the choice of c imposes the condition $2\left(1 - \frac{\beta_1}{|\mathcal{N}|}\right) < p$, which holds since we are assuming $p \leq \frac{1}{2}$ and $\beta_1 > \frac{3}{4}|\mathcal{N}|$.)

Since $b < \frac{1}{2}$, the large deviations result stated above gives

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Thus, x will have at most $(1-c)|\mathcal{N}|$ 1's in its neighborhood. By the choice of c, $(1-c)|\mathcal{N}| < \beta_1$. Thus, x will not change state at the next time step. It follows that no 0 will become a 1.

Now consider a site $y \in B_2(0, r\rho)$ that is in state 1. Pick b and C so that

$$\frac{2\delta_1}{|\mathcal{N}|} < 2C < b < \frac{1}{2}.$$

Case 1b : $p \geq \frac{1}{2}$, $\beta_1 > \frac{(1+p)}{2}|\mathcal{N}|$, and $\delta_1 < \frac{1}{4}|\mathcal{N}|$. Assume $C < \frac{1-p}{2}$. (Note that the choice of C imposes the condition $\frac{2\delta_1}{|\mathcal{N}|} < 1 - p$, which holds since we are assuming $\delta_1 < \frac{1}{4}|\mathcal{N}|$ and $p \geq \frac{1}{2}$.)

Arguing as we did in Case 1a above, we see that y will have at least $C|\mathcal{N}|$ $1's$ in its neighborhood and since $C|\mathcal{N}| > \delta_1$, no 1 will become a 0.

Case 2b : If $p \leq \frac{1}{2}$, $\beta_1 > \frac{3}{4}|\mathcal{N}|$, and $\delta_1 < \frac{p}{2}|\mathcal{N}|$. Assume $C < \frac{p}{2}$. (Note that the choice of C imposes the condition $\frac{2\delta_1}{|\mathcal{N}|} < p$, which holds since we are assuming $\delta_1 < \frac{p|\mathcal{N}|}{2}$.)

Arguing as we did in Case 2a above, we see that y will have at least $C|\mathcal{N}|$ $1's$ in its neighborhood and since $C|\mathcal{N}| > \delta_1$, no 1 will become a 0. $\square$

We point out that as the density of the initial state moves away from $\frac{1}{2}$, the restriction on the rules for which we have shown the sites never change becomes more strict. Thus, starting, from density $\frac{1}{2}$, as Durrett did in the TVA case, gives the result on the largest set of rules.

Let us briefly mention what the above says about finite ranges. Fix a finite range ρ. Let B be a box with side length n, where n is a large integer that depends on ρ. The previous theorem says that with high probability say $1 - \epsilon$, ($\epsilon > 0$ and depends on n), none of the sites in B will ever change state. By the ergodic theorem (see Chapter 6 of [Dur1]) with probability one at most proportion ϵ of the sites in the lattice ever change.

For LtL rules near the AVA, the analogue to the previous theorem is:

Theorem 4.1.3. Let B_k be the event that all sites x with $||x||_2 \leq k$ never change. Suppose the system starts from product measure with density p and that

$$\beta_1 = 0 \text{ and } \delta_1 = 1.$$

If $p \geq \frac{1}{2}$, $\delta_2 > \frac{(1+p)}{2}|\mathcal{N}|$, and $\beta_2 < \frac{1}{4}|\mathcal{N}|$, or if $p \leq \frac{1}{2}$, $\delta_2 > \frac{3}{4}|\mathcal{N}|$, and $\beta_2 < \frac{p}{2}|\mathcal{N}|$ then for all k, $P(B_k) \to 1$ as $\rho \to \infty$.

The proof is essentially the same as that for Theorem 4.1.2 except, in this case, the role of β_1 is played by δ_2 and that of δ_1 is played by β_2. For completeness, we illustrate these differences.

Proof of Theorem 4.1.3. Consider a site $x \in B_2(0, r\rho)$ that is in state 1. Pick b and c so that

$$2(1 - \frac{\delta_2}{|\mathcal{N}|}) < 2c < b < \frac{1}{2}.$$

Case 1a : $p \geq \frac{1}{2}$, $\delta_2 > \frac{(1+p)}{2}|\mathcal{N}|$, and $\beta_2 < \frac{1}{4}|\mathcal{N}|$**.** Assume $c < \frac{1-p}{2}$. (Note that the choice of c forces $2\left(1 - \frac{\delta_2}{|\mathcal{N}|}\right) < 1 - p$, which holds since we are assuming $\delta_2 > \frac{(1+p)}{2}|\mathcal{N}|$.)

As we found in the proof of the Theorem 4.1.2, P(all sites in $B_2(0, r\rho)$ have at least $c|\mathcal{N}|$ neighbors that are 0's) $\rightarrow 1$ as $\rho \rightarrow \infty$. Thus, x will have at most $(1 - c)|\mathcal{N}|$ 1's in its neighborhood. By the choice of c, $(1 - c)|\mathcal{N}| < \delta_2$. Thus, no 1 will become a 0.

Case 2a : $p \leq \frac{1}{2}$, $\delta_2 > \frac{3}{4}|\mathcal{N}|$, and $\beta_2 < \frac{p}{2}|\mathcal{N}|$**.** Assume $c < \frac{p}{2}$. (Note that the choice of c imposes the condition $2\left(1 - \frac{\delta_2}{|\mathcal{N}|}\right) < p$, which holds since we are assuming $p \leq \frac{1}{2}$ and $\delta_2 > \frac{3}{4}|\mathcal{N}|$.)

As we found in the proof of the Theorem 4.1.2, P(all sites in $B_2(0, r\rho)$ have at least $c|\mathcal{N}|$ neighbors that are 0's) $\rightarrow 1$ as $\rho \rightarrow \infty$. Thus, x will have at most $(1 - c)|\mathcal{N}|$ 1's in its neighborhood. By the choice of c, $(1 - c)|\mathcal{N}| < \delta_2$. Thus, no 1 will become a 0.

Now consider a site $y \in B_2(0, r\rho)$ that is in state 0. Pick b and C so that

$$\frac{2\beta_2}{|\mathcal{N}|} < 2C < b < \frac{1}{2}.$$

Case 1b : $p \geq \frac{1}{2}$, $\delta_2 > \frac{(1+p)}{2}|\mathcal{N}|$, and $\beta_2 < \frac{1}{4}|\mathcal{N}|$. Assume $C < \frac{1-p}{2}$. (Note that the choice of C imposes the condition $\frac{2\beta_2}{|\mathcal{N}|} < 1 - p$, which holds since we are assuming $\beta_2 < \frac{1}{4}|\mathcal{N}|$ and $\boldsymbol{p} \geq \frac{1}{2}$.)

Arguing as we did in Case 1a above, we see that y will have at least $C|\mathcal{N}|$ $1's$ in its neighborhood and since $C|\mathcal{N}| > \beta_2$, no 1 will become a 0.

Case 2b : $p \leq \frac{1}{2}$, $\delta_2 > \frac{3}{4}|\mathcal{N}|$, and $\beta_2 < \frac{p}{2}|\mathcal{N}|$. Assume $C < \frac{p}{2}$. (Note that the choice of C imposes the condition $\frac{2\beta_2}{|\mathcal{N}|} < p$, which holds since we are assuming $\beta_2 < \frac{p}{2}|\mathcal{N}|$.)

Arguing as we did in Case 2a above, we see that y will have at least $C|\mathcal{N}|$ $1's$ in its neighborhood and since $C|\mathcal{N}| > \beta_2$, no 1 will become a 0. □

The following theorem is another analogue of Theorem 4.1.2, for LtL rules near the AVA. In this case, however, all sites flip-flop every time step.

Theorem 4.1.4. Let C_k be the event that for each site x with $||x||_2 \leq k$, $\xi_t(x)$ has period 2 in t. Suppose the system starts from product measure with density p and that

$$\beta_1 = 0 \text{ and } \delta_1 = 1.$$

If $p \geq \frac{1}{2}$, $\beta_2 > \frac{(1+p)}{2}|\mathcal{N}|$, and $\delta_2 < (\frac{1-p}{2})|\mathcal{N}|$, or if $p \leq \frac{1}{2}$, $\beta_2 > (1 - \frac{p}{2})|\mathcal{N}|$and $\delta_2 < \frac{p}{2}|\mathcal{N}|$ then for all k, $P(C_k) \to 1$ as $\rho \to \infty$.

Proof of Theorem 4.1.4. Consider a site $x \in B_2(0, r\rho)$ that is in state 0. Pick b and c so that

$$2(1 - \tfrac{\beta_2}{|\mathcal{N}|}) < 2c < b < \tfrac{1}{2}.$$

Case 1a : $p \geq \frac{1}{2}$, $\beta_2 > \frac{(1+p)}{2}|\mathcal{N}|$, and $\delta_2 < (\frac{1-p}{2})|\mathcal{N}|$. Assume $c < \frac{1-p}{2}$. (Note that the choice of c imposes the condition $2(1 - \frac{\beta_2}{|\mathcal{N}|}) < 1 - p$, which holds since we are assuming $\beta_2 > \frac{(1+p)}{2}|\mathcal{N}|$.)

As we found in the proof of the Theorem 4.1.2, $P($all sites in $B_2(0, r\rho)$ have at least $c|\mathcal{N}|$ neighbors that are 0's$)$ $\to 1$ as $\rho \to \infty$. Thus, x will have at most $(1 - c)|\mathcal{N}|$ 1's in its neighborhood. By the choice of c, $(1 - c)|\mathcal{N}| < \beta_2$. Thus, x will become a 1. It follows that every site in $B_2(0, r\rho)$ that is in state 0 will become a 1 at the next time step.

Case 2a : If $p \leq \frac{1}{2}$, $\beta_2 > (1 - \frac{p}{2})|\mathcal{N}|$and $\delta_2 < \frac{p}{2}|\mathcal{N}|$. Assume $c < \frac{p}{2}$. (Note that the choice of c forces $2(1 - \frac{\beta_2}{|\mathcal{N}|}) < p$, which holds since we are assuming $p \leq \frac{1}{2}$ and $\beta_2 > (1 - \frac{p}{2})|\mathcal{N}|$.)

As we found in the proof of the Theorem 4.1.2, $P($all sites in $B_2(0, r\rho)$ have at least $c|\mathcal{N}|$ neighbors that are 0's$)$ $\rightarrow 1$ as $\rho \rightarrow \infty$. Thus, x will have at most $(1-c)|\mathcal{N}|$ 1's in its neighborhood. By the choice of c, $(1-c)|\mathcal{N}| < \beta_2$. Thus, x will become a 1. It follows that every site in $B_2(0, r\rho)$ that is in state 0 will become a 1 at the next time step.

Now consider a site $y \in B_2(0, r\rho)$ that is in state 1. Pick b and C so that

$$\frac{2\delta_2}{|\mathcal{N}|} < 2C < b < \frac{1}{2}.$$

Case 1b : $p \geq \frac{1}{2}$, $\beta_2 > \frac{(1+p)}{2}|\mathcal{N}|$, and $\delta_2 < (\frac{1-p}{2})|\mathcal{N}|$. Assume $C < \frac{1-p}{2}$. (Note that the choice of C imposes the condition $\frac{2\delta_2}{|\mathcal{N}|} < 1 - p$, which holds since we are assuming $\delta_2 < (\frac{1-p}{2})|\mathcal{N}|$.)

Arguing as we did in Case 1a above, we see that y will have at least $C|\mathcal{N}|$ $1's$ in its neighborhood and since $C|\mathcal{N}| > \delta_2$, y will become a 0. It follows that every site in $B_2(0, r\rho)$ that is in state 1 will become a 0 at the next time step.

Case 2b : If $p \leq \frac{1}{2}$, $\beta_2 > (1 - \frac{p}{2})|\mathcal{N}|$and $\delta_2 < \frac{p}{2}|\mathcal{N}|$. Assume $C < \frac{p}{2}$. (Note that the choice of C imposes the condition $\frac{2\delta_2}{|\mathcal{N}|} < p$, which holds since we are assuming $\delta_2 < \frac{p}{2}|\mathcal{N}|$.)

Arguing as we did in Case 1b above, we see that y will have at least $C|\mathcal{N}|$ $1's$ in its neighborhood and since $C|\mathcal{N}| > \delta_2$, y will become a 0. It follows that every site in $B_2(0, r\rho)$ that is in state 1 will become a 0 at the next time step. $\square$