2 The Hyper Picture

2.1 LtL Space

For a given range, ρ, each LtL rule can be represented by a point in the discrete four-dimensional subspace, $\mathcal{H}_\rho \subset \mathbb{Z}^4$, defined by

$$\mathcal{H}_\rho = \{(z_1, z_2, z_3, z_4) \in \mathbb{Z}^4 : 0 \leq z_1 \leq z_2 \leq (2\rho+1)^2 \text{ and } 0 \leq z_3 \leq z_4 \leq (2\rho+1)^2\}.$$

This holds since a range ρ LtL rule will be meaningful only if its parameters determine well-defined intervals. That is, the parameters must satisfy: $0 \leq \beta_1 \leq \beta_2 \leq (2\rho+1)^2$ and $0 \leq \delta_1 \leq \delta_2 \leq (2\rho+1)^2$. Additionally, each point in $\mathcal{H}_\rho$ represents a range ρ LtL rule. So we have both $\mathcal{H}_\rho \supset \{\text{range } \rho \text{ LtL rules}\}$ and $\mathcal{H}_\rho \subset \{\text{range } \rho \text{ LtL rules}\}$. Thus, $\mathcal{H}_\rho = \{\text{range } \rho \text{ LtL rules}\}$ which we refer to as LtL space. $\mathcal{H}_\rho$, or LtL space, is a proper subspace of the range ρ hypercube $\mathcal{H}^\rho \subset \mathbb{Z}^4$ defined by

$$\mathcal{H}^\rho = \{(z_1, z_2, z_3, z_4) \in \mathbb{Z}^4 : 0 \leq z_1, z_2 \leq (2\rho+1)^2 \text{ and } 0 \leq z_3, z_4 \leq (2\rho+1)^2\}.$$

For a fixed range ρ, there are $|\mathcal{H}_\rho| = (2k^2+3k+1)^2$ $(k = \frac{1}{2}(2\rho+1)^2)$ LtL rules. Since $|\mathcal{H}^\rho| = ((2\rho+1)^2+1)^4$, the LtL subspace represents

$$\frac{(2k^2+3k+1)^2}{((2\rho+1)^2+1)^4} \; (k = \frac{1}{2}(2\rho+1)^2)$$

of the points in $\mathcal{H}^\rho$. (Note that as $\rho \to \infty$ this proportion $\to \frac{1}{4}$.) Suppose we were to think of all of $\mathcal{H}^\rho$ as LtL space. Then each rule in $\mathcal{H}^\rho \setminus \mathcal{H}_\rho$ would result in global death. This is because neither birth nor survival would be possible for such a rule. (In other words, for such a rule after just one time step, regardless of the initial state, everything in the system would be dead.) That would be a little theorem (and a lot of progress) $-$ global death would occur for $\frac{3}{4}$ of the rules in $\mathcal{H}^\rho$.

We point out a couple of "issues" that come up because, according to LtL rules, live sites count themselves when deciding whether to survive. The first is that Life has LtL parameters, $(1,3,3,3,4)$, rather than the "usual" $(1,3,3,2,3)$. Another is that LtL rules with $\beta_2 = (2\rho+1)^2$ are equivalent to those with $\beta_2 = 4\rho(\rho+1)$, since it is impossible for a site that is a zero to see $(2\rho+1)^2$ $1's$ in its neighborhood. Similarly, rules with $\delta_1 = 0$ are equivalent to those with $\delta_1 = 1$ since a live site counts itself. However, if $\beta_1 = \beta_2 = (2\rho+1)^2$, then birth is impossible making such rules not redundant. Similarly, when $\delta_1 = \delta_2 = 0$ survival is impossible, so such rules also add something new to the set of meaningful LtL rules. Thus, in order to not be redundant, we restrict LtL parameters to $\beta_2 \leq 4\rho(\rho+1)$ and $1 \leq \delta_1$ except in the two cases mentioned above. This changes the total number of meaningful rules and removes some of the points in $\mathcal{H}_\rho$. However, for convenience, we still think of $\mathcal{H}_\rho$ as LtL space and use $(2k^2+3k+1)^2$ $(k = \frac{1}{2}(2\rho+1)^2)$ as the number of range ρ LtL rules. In most cases, this count is only used to find the proportion of rules with a particular limiting state as the range gets large; our restriction does not change this value.

2.2 Threshold-range scaling; a Riemannian approach

What happens if we vary the range and consider the respective hypercubes in terms of the limiting states of the rules at each point? Are there regions of the hypercubes which contain particular dynamics that rescale in a coherent manner? A natural approach that leads to an affirmative answer, is to rescale the rules' parameters in terms of their proportions of their neighborhoods as follows.

Embed $\mathcal{H}^\rho$ in the hypercube $\widetilde{\mathcal{H}}^\rho \subset \mathbb{R}^4$ defined by

$$\widetilde{\mathcal{H}}^\rho = \{(z_1, z_2, z_3, z_4) \in \mathbb{R}^4 : 0 \leq z_1, z_2 \leq (2\rho+1)^2 \text{ and } 0 \leq z_3, z_4 \leq (2\rho+1)^2\}.$$

Let $\mathcal{U}$ be the four-dimensional unit hypercube contained in $\mathbb{R}^4$. That is,

$$\mathcal{U} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : 0 \leq x_i \leq 1,\ i = 1, 2, 3, 4\}.$$

Define the mapping $f: \widetilde{\mathcal{H}}^{\rho} \to \mathcal{U}$ by,

$$f(z_1, z_2, z_3, z_4) = (x_1, x_2, x_3, x_4)$$

where

$$x_1 = \begin{cases} \frac{z_1 - 0.5}{(2\rho+1)^2} & \text{if } z_1 \neq 0 \\ 0 & \text{if } z_1 = 0 \end{cases}$$

$$x_2 = \begin{cases} \frac{z_2 + 0.5}{(2\rho+1)^2} & \text{if } z_2 \neq (2\rho+1)^2 \\ 1 & \text{if } z_2 = (2\rho+1)^2 \end{cases}$$

$$x_3 = \begin{cases} \frac{z_3 - 0.5}{(2\rho+1)^2} & \text{if } z_3 \neq 0 \\ 0 & \text{if } z_3 = 0 \end{cases}$$

$$x_4 = \begin{cases} \frac{z_4 + 0.5}{(2\rho+1)^2} & \text{if } z_4 \neq (2\rho+1)^2 \\ 1 & \text{if } z_4 = (2\rho+1)^2 \end{cases}$$

For a given range, ρ, this mapping, f, partitions the sides of $\mathcal{U}$ into $(2\rho+1)^2+1$ subintervals that are determined by the $(2\rho+1)^2+2$ points,

$$\{0, \frac{0.5}{(2\rho+1)^2}, \frac{1.5}{(2\rho+1)^2}, \frac{2.5}{(2\rho+1)^2}, \ldots, \frac{(2\rho+1)^2-0.5}{(2\rho+1)^2}, 1\}.$$

For example, in range one the sides are divided approximately as:

In this way, the hypercube is partitioned into $((2\rho+1)^2+1)^4$ hypercubes, each of which has hypervolume

$$\frac{c}{(2\rho+1)^8}$$

where $c = 1$ if it is an interior hypercube, and $c = \frac{1}{16}$ if it lies along one of the boundaries of $\mathcal{U}$. As the range tends to ∞, the number of hypercubes in the partition tends to ∞, and their hypervolumes tend to 0. Each point in the unit hypercube represents an LtL rule in the *threshold-range scaling limit*, $\rho \to \infty$ (see [FGG1]).

f maps rules into the unit hypercube, but how do we rescale rules from one range to another? Or, what if we want to begin with any point of $\mathcal{U}$ and determine what rule it represents in a particular range? Keeping with the scaling scheme above, we multiply the coordinates of the points in $\mathcal{U}$ by the total size of the neighbor set. That is, let $x = (x_1, x_2, x_3, x_4) \in \mathcal{U}$. Then, in range ρ, the rule is approximately,

$$(\beta_1, \beta_2, \delta_1, \delta_2) = (\lceil x_1(2\rho+1)^2 \rceil, \lfloor x_2(2\rho+1)^2 \rfloor, \lceil x_3(2\rho+1)^2 \rceil, \lfloor x_4(2\rho+1)^2 \rfloor).$$

We will see in what follows that this scheme is both good and bad, depending on the region of parameter space with which we are dealing and, in some cases, on the range in which we are working. We note, however, that even in regions where the scheme is good, it is often necessary to slightly vary one or more of the parameters if one desires to find very similar dynamics through the ranges. An example of when this scheme is not very accurate is when all four parameters are equal or differ by just one or so. In those cases, the dynamics rescale in a linear, rather than quadratic, fashion. In fact, there is a linear scaling scheme that gives nontrivial and consistent limits in those cases where scaling with respect to area does not. We discuss this further in Sections 4.4 and 7.1.

Although neither the linear nor the quadratic scaling scheme is perfect, the quadratic scheme seems to do the best job. The fact that this set of rules is rather difficult to precisely rescale further illustrates the nonlinearities involved. As we will see, some dynamics depend mostly on a single parameter, while others depend intrinsically on all four.