THE KINETIC THEORY OF SIMPLE REACTING SPHERES: I. GLOBAL EXISTENCE RESULT IN A DILUTE-GAS CASE

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Dedicated to George Stell

Abstract. Existence of global in time, spatially inhomogeneous, and L^1-renormalized solutions is proven for the model of simple reacting spheres under the assumptions that initially the system has a finite total mass, energy, and entropy.

1. Introduction

The kinetic theory of simple reacting spheres (SRS) had been proposed by Marron [1] and further developed by Xystris and Dahler [2]. In the model the molecules behave as if they were single mass points with two internal states of excitation. Collisions may alter the internal states (this occurs when the kinetic energy associated with the reactive motion exceeds the activation energy) but can not transfer mass from one molecule to another. Reactive and non-reactive collision events are considered to be hard spheres–like. I start by considering a four component mixture A, B, A^*, B^*, and the chemical reaction of the type

$$A + B \rightleftharpoons A^* + B^*. \tag{1.1}$$

Here, A^* and B^* are the distinct species from A and B. In the paper I use the indices 1, 2, 3, and 4 for the particles A, B, A^*, and B^*, respectively. I assume **no** net mass transfer in reactive collisions; this implies $m_1 = m_3$ and $m_2 = m_4$, where m_i denotes the mass of the i-th particle, $i = 1, \ldots, 4$. Reactions take place when the reactive particles are separated by a distance $\sigma_{12} = \frac{1}{2}(d_1 + d_2)$, where d_i denotes the diameter of the i-th particle. Since in the SRS model reactions do not change diameters of the particles, $d_1 = d_3$ and $d_2 = d_4$. The last set of equalities also implies that $\sigma_{34} = \frac{1}{2}(d_3 + d_4) = \sigma_{12}$. I note that by not allowing the hard sphere diameter to change upon reaction one avoids complications of producing overlapping configurations (see, [3]).

Key words and phrases. Kinetic theory of gas mixtures, chemical reactions, reacting mixtures, simple reacting spheres, hard-sphere systems, initial value problem.

In contrast to some more advanced models of chemical reactions considered in the literature (see e.g., the references in [1] for multiply reacting rigid spheres (MIRS) models), internal degrees variables do not appear explicitly in the collisional integrals of the kinetic equation based on the SRS model. The SRS, being a natural extension of the hard–sphere collisional model, reduces itself to the Enskog theory when the chemical reactions are turned off. Furthermore, in the dilute–gas limit it provides an interesting kinetic model of chemical reactions that has not been considered before.

In a series of papers C. P. Grünfeld and E. Georgescu ([4], [5]) consider a general class of Boltzmann-like kinetic equations with multiple inelastic collisions, where they prove existence and uniqueness of vacuum-type solutions for small initial data. M. Groppi, A. Rossani, and G. Spiga in [6] and [7] formally analyze various kinetic theories of chemically reacting gases , including gas-photon interactions. They show existence of an H-function and described possible equilibrium solutions. Their results are based on the micro-reversibility conditions that relate the differential cross-section scattering kernels before and after reactive collisions. In the case of SRS, however, the reacting molecules behave like hard spheres before and after reactive collisions. Thus, the micro-reversibility conditions reduce themselves to the symmetries of the separation distances $\sigma_{12} = \sigma_{34}$ and the steric factors $\beta_{ij} = \beta_{ji}$ (see, (2.11)–(2.12)).

After introducing a general model in Section 2, I consider, in Section 3, important physical properties of the dilute SRS kinetic equations. They will play a fundamental role in proving existence of renormalized solutions (see, [8] for a single specie Boltzmann equation), global in time, and under the assumptions that, initially, the system has a finite total mass, energy, and entropy. Section 4 contains the existence result and its proof. This is the first part of a series of papers on kinetic equations of simple reacting spheres. The rigorous results concerning asymptotical behavior, convergence to equilibrium, passage to hydrodynamics, and the case of the dense-gas SRS kinetic equations will appear in forthcoming papers.

2. The SRS kinetic system

Following [9], for each i $(i = 1, \ldots, 4)$, $f_i(t, x, v)$ denotes the one-particle distribution function of the ith component of the reactive mixture. The function $f_i(t, x, v)$, which changes in time due to free streaming and collisions (both elastic and reactive), represents at time t the number density of particles at point x with velocity v.

In the case of elastic encounters between a pair of particles from species i and s, the initial velocities v, w take post–collisional values

$$v' = v - 2\frac{\mu_{is}}{m_i}\epsilon\langle\epsilon, v-w\rangle, \quad w' = w + 2\frac{\mu_{is}}{m_s}\epsilon\langle\epsilon, v-w\rangle. \tag{2.1}$$

Here, $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^3$, ϵ is a vector along the line passing through the centers of the spheres at the moment of impact, i.e., $\epsilon \in \mathbb{S}^2_+ = \{\epsilon \in \mathbb{R}^3 : |\epsilon| = 1, \langle\epsilon, v-w\rangle \geq 0\}$. Also, $\mu_{is} = \frac{m_i m_s}{m_i+m_s}$ is the reduced mass of the colliding pair, where m_i and m_s are the masses of particles from i-th and s-th species, respectively.

Finally, let us note that conditions $m_1 = m_3$ and $m_2 = m_4$ imply $\mu_{12} = \mu_{34}$. This property is crucial to prove the main results in this work.

For the reactive collision between particles of species i and s to occur $(i, s = 1, \ldots, 4)$, the kinetic energy associated with the relative motion along the line of centers must exceed the activation energy γ_i (defined below),

$$\frac{1}{2}\mu_{is}\big(\langle\epsilon, v-w\rangle\big)^2 \geq \gamma_i, \tag{2.2}$$

with ϵ having the same meaning as above. In the case of the reaction $A + B \rightarrow A^* + B^*$ the velocities v, w take their post–reactive values

$$v^\ddagger = v - \frac{\mu_{12}}{m_1}\epsilon\Big[\langle\epsilon, v-w\rangle - \alpha^-\Big], \quad w^\ddagger = w + \frac{\mu_{12}}{m_2}\epsilon\Big[\langle\epsilon, v-w\rangle - \alpha^-\Big], \tag{2.3}$$

with $\alpha^- = \sqrt{\big(\langle\epsilon, v-w\rangle\big)^2 - 2E_{abs}/\mu_{12}}$ and, E_{abs}, the energy absorbed by the internal degrees of freedom. The absorbed energy E_{abs} has the property

$$E_{abs} = E_3 + E_4 - E_1 - E_2 > 0, \tag{2.4}$$

where $E_i > 0$, $i = 1, \ldots 4$, is the energy of i-th particle associated with its internal degrees of freedom.

Now, in order to complete the definition of the model, the activation energies γ_1, γ_2 for A and B are chosen to satisfy $\gamma_1 \geq E_{abs} > 0$, and by symmetry, $\gamma_2 = \gamma_1$.

For the inverse reaction, $A^* + B^* \rightarrow A + B$, the post–reactive velocities are given by

$$v^\dagger = v - \frac{\mu_{34}}{m_3}\epsilon\Big[\langle\epsilon, v-w\rangle - \alpha^+\Big], \quad w^\dagger = w + \frac{\mu_{34}}{m_4}\epsilon\Big[\langle\epsilon, v-w\rangle - \alpha^+\Big], \tag{2.5}$$

with $\alpha^+ = \sqrt{\big(\langle\epsilon, v-w\rangle\big)^2 + 2E_{abs}/\mu_{34}}$, and the activation energies for A^* and B^*, $\gamma_3 = \gamma_1 - E_{abs}$ and, as before, $\gamma_4 = \gamma_3$.

The pairs of velocities in (2.3) and (2.5) satisfy conservation of the momentum

$$m_1 v + m_2 w = m_1 v^{\ddagger} + m_2 w^{\ddagger} = m_3 v^{\ddagger} + m_4 w^{\ddagger}, \quad m_3 v + m_4 w = m_3 v^{\dagger} + m_4 w^{\dagger} = m_1 v^{\dagger} + m_2 w^{\dagger}; \tag{2.6}$$

they do not, however, obey conservation of the kinetic energy. A part of kinetic energy is exchanged with the energy absorbed by the internal states. The following equalities hold:

$$\begin{aligned} m_1 v^2 + m_2 w^2 &= m_1 v^{\ddagger 2} + m_2 w^{\ddagger 2} + 2E_{abs} = m_3 v^{\ddagger 2} + m_4 w^{\ddagger 2} + 2E_{abs}, \\ m_3 v^2 + m_4 w^2 &= m_3 v^{\dagger 2} + m_4 w^{\dagger 2} - 2E_{abs} = m_1 v^{\dagger 2} + m_2 w^{\dagger 2} - 2E_{abs}. \end{aligned} \tag{2.7}$$

Also, it is easy to show that the relatives velocities before and after reactions, i.e., $V = v - w$, $V^{\ddagger} = v^{\ddagger} - w^{\ddagger}$, and $V^{\dagger} = v^{\dagger} - w^{\dagger}$, respectively, satisfy the identities

$$V^{\ddagger 2} = V^2 - \frac{2E_{abs}}{\mu_{12}}, \qquad V^{\dagger 2} = V^2 + \frac{2E_{abs}}{\mu_{34}}. \tag{2.8}$$

Finally, the reactive collisions $A + B \rightleftharpoons A^* + B^*$ can be also represented in the form $i + j \to k + l$, where the set of indices (i, j, k, l) can be enumerated:

$$(1, 2, 3, 4), \quad (2, 1, 4, 3), \quad (3, 4, 1, 2), \quad (4, 3, 2, 1) \tag{2.9}$$

Now, for $i = 1, \ldots, 4$, the SRS kinetic system can be expressed as follows

$$\frac{\partial f_i}{\partial t} + v \frac{\partial f_i}{\partial x} = J_i^E + J_i^R, \tag{2.10}$$

with

$$\begin{aligned} J_i^E = \sum_{s=1}^{4} \Bigg\{ \sigma_{is}^2 \iint\limits_{\mathbb{R}^3 \times \mathbb{S}^2} \Big[f_{is}^{(2)}(t, x, v', x - \sigma_{is}\epsilon, w') - f_{is}^{(2)}(t, x, v, x + \sigma_{is}\epsilon, w) \Big] \Theta\left(\langle \epsilon, v - w \rangle\right) \langle \epsilon, v - w \rangle \, d\epsilon dw \Bigg\} \\ - \beta_{ij}\sigma_{ij}^2 \iint\limits_{\mathbb{R}^3 \times \mathbb{S}^2} \Big[f_{ij}^{(2)}(t, x, v', x - \sigma_{ij}\epsilon, w') - f_{ij}^{(2)}(t, x, v, x + \sigma_{ij}\epsilon, w) \Big] \Theta\left(\langle \epsilon, v - w \rangle - \Gamma_{ij}\right) \langle \epsilon, v - w \rangle \, d\epsilon dw, \end{aligned} \tag{2.11}$$

and

$$J_i^R = \beta_{ij}\sigma_{ij}^2 \iint\limits_{\mathbb{R}^3 \times \mathbb{S}^2} \Big[f_{kl}^{(2)}(t, x, v_{ij}^{\odot}, x - \sigma_{ij}\epsilon, w_{ij}^{\odot}) - f_{ij}^{(2)}(t, x, v, x + \sigma_{ij}\epsilon, w) \Big] \Theta\left(\langle \epsilon, v - w \rangle - \Gamma_{ij}\right) \langle \epsilon, v - w \rangle \, d\epsilon dw. \tag{2.12}$$

Here, the function $f_{is}^{(2)}$ approximates the density of pairs of particles in collisional configurations, $0 \le \beta_{ij} < 1$ is the steric factor for reactive collisions between species i and j, $\Gamma_{ij} = \sqrt{2\gamma_i/\mu_{ij}}$, and Θ is the Heaviside step function. The prime velocities in (2.11) are given in (2.1). The pair of velocities $(v_i^{\odot}, v_j^{\odot})$ refers to post-reactive velocities described either in (2.3) or (2.5), i.e., $(v_{ij}^{\odot}, w_{ij}^{\odot}) = (v^{\ddagger}, w^{\ddagger})$ for $i, j = 1, 2$, and $(v_{ij}^{\odot}, w_{ij}^{\odot}) = (v^{\dagger}, w^{\dagger})$ for $i, j = 3, 4$. Also, the index pairs (i, j) and (k, l) appearing in (2.11)-(2.12) are associated with the set of indices (i, j, k, l) specified in (2.9).

The first term of (2.11) is a hard-spheres collision operator with the usual pre-collisional range of integration, while the second term of (2.11) singles out those pre-collisional states that are energetic enough to result in reaction. The collision operator in (2.12) is purely reactive.

When the steric factors $\beta_{ij} = 0$, i.e., there are no reactive collisions, and $f_{is}^{(2)}$ is the exact two–particle distribution function, system (2.10)–(2.12) becomes the exact first BBGKY hierarchy system for a four component hard–spheres mixture (with the diameters and masses satisfying $d_1 = d_3$, $d_2 = d_4$ and $m_1 = m_3$, $m_2 = m_4$, respectively). As in the kinetic theory of non–reactive mixtures, different ways in which one approximates the two–particle distribution function $f_{ij}^{(2)}$ give rise to different kinetic equations. For this purpose it is convenient to write $f_{ij}^{(2)}$ in form of the closure relation

$$f_{ij}^{(2)}(t, x, v, y, w) = Y_{ij}\left(t, x, v, y, w \,|\, \{\Lambda_i f_i\}\right) f_i(t, x, v) f_j(t, y, w), \tag{2.13}$$

where Y_{ij} is assumed to be given, for each i and j and for each fixed $t \geq 0$, $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4)$ is an (possibly nonlinear) operator acting on (f_1, f_2, f_3, f_4), typically through one or more velocity moments, In [10] various forms of Λ and the resulting kinetic equations were considered. For example, in the case of the revised Enskog system for non-reactive mixtures (see, [11] and [12]) $Y_{ij} = Yij^{RET}$ has the form

$$Y_{ij}^{RET} = g_{ij}^{(2)}\left(x_1, x_2 \,|\, \{n_i(t, \cdot)\}\right) \tag{2.14}$$

where $n_i(t, x) = \int_{\mathbb{R}^3} f_i(t, x, v)dv$ is the local number density of the component i and $g_{ij}^{(2)}$ is the pair correlation function for a **non-uniform** system at equilibrium with the local densities $n_i(t, x)$. The notation $g_{ij}^{(2)}(x_1, x_2 \,|\, \{n_i(t, \cdot)\})$ indicates that $g_{ij}^{(2)}$ is a functional of the local densities n_i.

In this work, I will be concerned only with the dilute-gas limit of the system (2.10)–(2.12). Formally at least, one can show that when $\sigma_{ij} \to 0$, $n_i \to 0$, with $n_i \sigma_{ij}^2 \to \mathit{const} \neq 0$ and $n_i \sigma_{ij}^3 \to 0$, then $g_{ij}^{(2)} \to 1$ in (2.14). Another (more ad hoc) way to obtain the system of reactive kinetic equations for dilute-gas regime is to take $Y_{ij} \equiv 1$, for $i = 1, \ldots, 4$ and assume that the change of $f_i(t, x, v)$ over a length σ_{ij}, for arbitrary t and v, is negligible (resulting in $f_i(t, x, v) \approx f_i(t, x + \sigma_{ij}\epsilon, v)$).

Let us notice that in the dilute-gas limit the system (2.10)–(2.12), with $\beta_{ij} = 0$ (no reactive collisions), becomes the Boltzmann system for hard–spheres mixture. This fact becomes even more important if one realizes that the cross sections of gas phase reactions are usually smaller as compared to the non-reactive collisions. This way the reactive collision terms can be considered as perturbative corrections to non-reactive collisional terms.

3. Properties of the dilute SRS kinetic system

The main result of this work is the global existence theorem for the dilute-gas system

$$\frac{\partial f_i}{\partial t} + v\frac{\partial f_i}{\partial x} = J_i^E + J_i^R, \quad f_i(0,x,v) = f_{i0}(x,v), \quad i = 1,\dots,4, \quad (x,v) \in \Omega \times \mathbb{R}^3 \tag{3.1}$$

with

$$\begin{aligned} J_i^E = &\sum_{s=1}^{4}\left\{\sigma_{is}^2 \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} \Big[f_i(t,x,v')f_s(t,x,w') - f_i(t,x,v)f_s(t,x,w)\Big]\,\Theta\left(\langle\epsilon, v-w\rangle\right)\langle\epsilon, v-w\rangle\, d\epsilon dw\right\} \\ &- \beta_{ij}\sigma_{ij}^2 \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} \Big[f_i(t,x,v')f_s(t,x,w') - f_i(t,x,v)f_s(t,x,w)\Big]\Theta\left(\langle\epsilon, v-w\rangle - \Gamma_{ij}\right)\langle\epsilon, v-w\rangle\, d\epsilon dw, \end{aligned} \tag{3.2}$$

and

$$J_i^R = \beta_{ij}\sigma_{ij}^2 \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} \Big[f_k(t,x,v_{ij}^{\odot})f_l(t,x,w_{ij}^{\odot}) - f_i(t,x,v)f_j(t,x,w)\Big]\Theta\left(\langle\epsilon, v-w\rangle - \Gamma_{ij}\right)\langle\epsilon, v-w\rangle\, d\epsilon dw, \tag{3.3}$$

where f_{i0}, $i = 1,\dots,4$ are suitable nonnegative initial conditions that will be defined later. The gas mixture is confined in $\Omega \subseteq \mathbb{R}^3$. I consider two choices for the set Ω:, $\Omega = \mathbb{R}^3$, or Ω being a 3-dimensional torus $[0,L]^3$, $L > 0$. The latter choice corresponds to case of the periodic boundary conditions on $[0,L]^3$.

The following properties of (3.1)–(3.3) are crucial in proving the existence result.

Proposition 3.1. *Assume that $\beta_{ij} = \beta_{ji}$ for $(i,j) \in \{(1,2),(2,1),(3,4),(4,3)\}$. Then for ϕ_i measurable on $\Omega \times \mathbb{R}^3$ and $f_i \in C_0(\Omega \times \mathbb{R}^3)$, $i = 1,\dots,4$,*

$$\begin{aligned} \sum_{i=1}^{4}\int\limits_{\mathbb{R}^3} \phi_i J_i^E\, dv = \sum_{i=1}^{4}\sum_{s=1}^{4}\sigma_{is}^2 \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} &\left[\phi_i(x,v) + \phi_s(x,w) - \phi_i(x,v') - \phi_s(x,w')\right] \times \\ &\left[f_i(v')f_s(w') - f_i(v)f_s(w)\right]\Theta(\langle\epsilon, v-w\rangle)\langle\epsilon, v-w\rangle\Xi_{is}\, d\epsilon dw dv, \end{aligned} \tag{3.4}$$

$$\sum_{i=1}^{4}\int_{\mathbb{R}^3}\phi_i J_i^R\,dv = \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}\left[\beta_{12}\sigma_{12}^2\phi_1(x,v)+\beta_{21}\sigma_{21}^2\phi_2(x,w)-\beta_{34}\sigma_{34}^2\phi_3(x,v^{\ddagger})-\beta_{43}\sigma_{43}^2\phi_4(x,w^{\ddagger})\right]\times \\ \left[f_3(x,v^{\ddagger})f_4(x,w^{\ddagger})-f_1(x,v)f_2(x,w)\right]\Theta(\langle\epsilon,v-w\rangle-\Gamma_{12})\langle\epsilon,v-w\rangle\,d\epsilon dw dv, \tag{3.5}$$

and

$$\sum_{i=1}^{4}\int_{\mathbb{R}^3}\phi_i J_i^R\,dv = \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}\left[\beta_{34}\sigma_{34}^2\phi_3(x,v)+\beta_{43}\sigma_{43}^2\phi_4(x,w)-\beta_{12}\sigma_{12}^2\phi_1(x,v^{\dagger})-\beta_{21}\sigma_{21}^2\phi_2(x,v^{\dagger})\right]\times \\ \left[f_1(x,v^{\dagger})f_2(x,w^{\dagger})-f_3(x,v)f_4(x,w)\right]\Theta(\langle\epsilon,v-w\rangle-\Gamma_{34})\langle\epsilon,v-w\rangle\,d\epsilon dw dv, \tag{3.6}$$

where X_{is}, appearing in (3.4), *is given by*

$$\Xi_{is} = \begin{cases} \frac{1}{2}\Theta(\langle\epsilon,v-w\rangle-\Gamma_{is})+\frac{1}{2}(1-\beta_{is})\Theta(\Gamma_{is}-\langle\epsilon,v-w\rangle), & \text{if } (i,s)\in I;\\ \frac{1}{4}\Theta(\langle\epsilon,v-w\rangle), & \text{if } i=s;\\ \frac{1}{2}\Theta(\langle\epsilon,v-w\rangle), & \text{otherwise,} \end{cases} \tag{3.7}$$

with $I=\{(1,2),(2,1),(3,4),(4,3)\}$.

The post-collisional velocities, v' and w' are given in (2.1), *while the post-reactive velocities, $v^{\ddagger}$, $w^{\ddagger}$ and $v^{\dagger}$, $w^{\dagger}$, are given in* (2.3) *and* (2.5), *respectively.*

Proof. The proof of (3.4) is standard, see, for example, [13] for single specie treatment. The proof for mixture gases is similar: it is based on the fact that the absolute value of the Jacobians of the transformations $(v,w)\mapsto(v',w')$ and $(v,w)\mapsto(w,v)$ are one, together with the identity $\langle\epsilon,v-w\rangle=\langle-\epsilon,w-v\rangle$. The change of variables, $(v,w)\mapsto(v',w')$, $(v,w)\mapsto(w,v)$, and $\epsilon\mapsto-\epsilon$, together with the fact that $\beta_{is}=\beta_{si}$, results in (3.4). The multiplicative factor X_{is} comes from the fact that second term of the non-reactive collisional integral (3.2), with β_{ij} in front of it, singles out those pre-collisional states that are energetic enough to result in the reaction, and thus preventing double counting of the events in the collisional integrals (3.2)–(3.3).

In order to prove (3.5) and (3.6) one needs the following lemma.

Lemma 3.1. *For fixed ϵ, the Jacobians of the transformations $(v,w)\mapsto(v^{\dagger},w^{\dagger})$ and $(v,w)\mapsto(v^{\ddagger},w^{\ddagger})$ are given by $\langle\epsilon,v-w\rangle/\alpha^{+}$ and $\langle\epsilon,v-w\rangle/\alpha^{-}$, respectively. Furthermore, $\langle\epsilon,v^{\dagger}-w^{\dagger}\rangle=\alpha^{+}$ and $\langle\epsilon,v^{\ddagger}-w^{\ddagger}\rangle=\alpha^{-}$.*

Proof of Lemma 3.1. If $J(v^\dagger, w^\dagger; v, w)$ and $J(G_{34}^\dagger, V^\dagger; G_{43}, V)$ denote the Jacobians of the transformations $(v,w) \mapsto (v^\dagger, w^\dagger)$ and $(G_{34}, V) \mapsto (G_{34}^\dagger, V^\dagger)$, respectively, where

$$\begin{aligned} G_{34}(v,w) &= m_3 v + m_4 w, && \text{(the velocity of the center of mass before reaction)} \\ V(v,w) &= v - w, && \text{(the relative velocity before reaction)} \\ G_{34}^\dagger(v^\dagger, w^\dagger) &= m_3 v^\dagger + m_4 w^\dagger, && \text{(the velocity of the center of mass after reaction)} \\ V^\dagger(v^\dagger, w^\dagger) &= v^\dagger - w^\dagger, && \text{(the relative velocity after reaction)} \end{aligned} \tag{3.8}$$

then the following equality holds

$$J(v^\dagger, w^\dagger; v, w) = J(v^\dagger, w^\dagger; G_{34}^\dagger, V^\dagger) J(G_{34}^\dagger, V^\dagger; G_{34}, V) J(G_{34}, V; v, w) = J(G_{34}^\dagger, V^\dagger; G_{34}, V). \tag{3.9}$$

Note that $J(v^\dagger, w^\dagger; G_{34}^\dagger, V^\dagger) = 1/J(G_{34}, V; v, w)$. Next, the conservation of momentum before and after reaction implies that $G_{34}^\dagger = G_{34}$ (see, (2.6)), and thus

$$J(G_{34}^\dagger, V^\dagger; G_{34}, V) = J(V^\dagger, V), \tag{3.10}$$

where $J(V^\dagger, V)$ is the Jacobian of the transformation $V \mapsto V^\dagger$ given by

$$V^\dagger = V - \epsilon\left[\langle \epsilon, V\rangle - \alpha^+\right] = V - \epsilon\left[\langle \epsilon, V\rangle - \sqrt{\langle \epsilon, V\rangle^2 + \frac{2E_{abs}}{\mu_{34}}}\right] \tag{3.11}$$

The value of $J(V^\dagger, V)$ is $\langle \epsilon, V\rangle / \sqrt{\langle \epsilon, V\rangle^2 + \frac{2E_{abs}}{\mu_{34}}}$, This shows that $J(v^\dagger, w^\dagger; v, w) = \langle \epsilon, v - w\rangle/\alpha^+$. The proof that $J(v^\ddagger, w^\ddagger; v, w) = \langle \epsilon, v - w\rangle/\alpha^-$ follows the same arguments as above. Finally, using the definitions (2.3) and (2.5) together with simple algebra one obtains the identities $\langle \epsilon, v^\dagger - w^\dagger\rangle = \alpha^+$ and $\langle \epsilon, v^\ddagger - w^\ddagger\rangle = \alpha^-$. This completes the proof of Lemma 3.1. □

Next, I consider the integrals

$$\int_{\mathbb{R}^3} \phi_1 J_1^R \, dv = \beta_{12}\sigma_{12}^2 \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \phi_1(v)\left[f_3(v^\ddagger) f_4(w^\ddagger) - f_1(v) f_2(w)\right] \langle \epsilon, v - w\rangle \Theta(\langle \epsilon, v - w\rangle - \Gamma_{12}) \, d\epsilon dw dv, \tag{3.12}$$

$$\int_{\mathbb{R}^3} \phi_2 J_2^R \, dv = \beta_{21}\sigma_{21}^2 \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \phi_2(v)\left[f_4(v^\ddagger) f_3(w^\ddagger) - f_2(v) f_1(w)\right] \langle \epsilon, v - w\rangle \Theta(\langle \epsilon, v - w\rangle - \Gamma_{12}) \, d\epsilon dw dv, \tag{3.13}$$

$$\int_{\mathbb{R}^3} \phi_3 J_3^R \, dv = \beta_{34}\sigma_{34}^2 \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \phi_3(v)\left[f_1(v^\dagger) f_2(w^\dagger) - f_3(v) f_4(w)\right] \langle \epsilon, v - w\rangle \Theta(\langle \epsilon, v - w\rangle - \Gamma_{34}) \, d\epsilon dw dv, \tag{3.14}$$

and

$$\int\limits_{\mathbb{R}^3} \phi_4 J_4^R \, dv = \beta_{43}\sigma_{43}^2 \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \phi_4(v) \left[f_2(v^\dagger)f_1(w^\dagger) - f_4(v)f_3(w)\right] \langle \epsilon, v-w\rangle \Theta(\langle \epsilon, v-w\rangle - \Gamma_{34}) \, d\epsilon dw dv, \quad (3.15)$$

appearing in the sum on the left hand side of (3.5). In (3.12)–(3.15), I also suppressed x dependence in ϕ_i and f_i. Changing the variables of integration in (3.14)–(3.15) from (v, w) to $(v^\dagger, w^\dagger)$ and using Lemma 3.1 one obtains

$$\int\limits_{\mathbb{R}^3} \phi_3 J_3^R \, dv = \beta_{34}\sigma_{34}^2 \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \phi_3(v) \left[f_1(v^\dagger)f_2(w^\dagger) - f_3(v)f_4(w)\right] \langle \epsilon, v^\dagger - w^\dagger\rangle \Theta(\langle \epsilon, v-w\rangle - \Gamma_{34}) \, d\epsilon dw^\dagger dv^\dagger \quad (3.14')$$

and

$$\int\limits_{\mathbb{R}^3} \phi_4 J_4^R \, dv = \beta_{43}\sigma_{43}^2 \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \phi_4(v) \left[f_2(v^\dagger)f_1(w^\dagger) - f_4(v)f_3(w)\right] \langle \epsilon, v^\dagger - w^\dagger\rangle \Theta(\langle \epsilon, v-w\rangle - \Gamma_{34}) \, d\epsilon dw^\dagger dv^\dagger . \quad (3.15')$$

Next, one notices that v and w (as the functions of $v^\dagger$, $w^\dagger$) become

$$\begin{aligned} v &= v^\dagger + \frac{\mu_{34}}{m_3}\epsilon \left[\langle \epsilon, v-w\rangle - \alpha^+\right] = v^\dagger - \frac{\mu_{12}}{m_1}\epsilon \left[\langle \epsilon, v^\dagger - w^\dagger\rangle - \langle \epsilon, v-w\rangle\right] \\ &= v^\dagger - \frac{\mu_{12}}{m_1}\epsilon \Bigg[\langle \epsilon, v^\dagger - w^\dagger\rangle - \Bigg(\underbrace{(\langle \epsilon, v-w\rangle)^2 + \frac{2Eabs}{\mu_{34}}}_{(\langle \epsilon, v^\dagger - w^\dagger\rangle)^2} - \frac{2Eabs}{\mu_{12}}\Bigg)^{\frac{1}{2}}\Bigg] \\ &= v^\dagger - \frac{\mu_{12}}{m_1}\epsilon \left[\langle \epsilon, v^\dagger - w^\dagger\rangle - \alpha^-(v^\dagger, w^\dagger)\right] = v^\ddagger(v^\dagger, w^\dagger) \end{aligned} \quad (3.16)$$

and

$$\begin{aligned} w &= w^\dagger - \frac{\mu_{34}}{m_3}\epsilon \left[\langle \epsilon, v-w\rangle - \alpha^+\right] = w^\dagger + \frac{\mu_{12}}{m_1}\epsilon \left[\langle \epsilon, v^\dagger - w^\dagger\rangle - \langle \epsilon, v-w\rangle\right] \\ &= w^\dagger + \frac{\mu_{12}}{m_1}\epsilon \Bigg[\langle \epsilon, v^\dagger - w^\dagger\rangle - \Bigg(\underbrace{(\langle \epsilon, v-w\rangle)^2 + \frac{2Eabs}{\mu_{34}}}_{(\langle \epsilon, v^\dagger - w^\dagger\rangle)^2} - \frac{2Eabs}{\mu_{12}}\Bigg)^{\frac{1}{2}}\Bigg] \\ &= w^\dagger + \frac{\mu_{12}}{m_1}\epsilon \left[\langle \epsilon, v^\dagger - w^\dagger\rangle - \alpha^-(v^\dagger, w^\dagger)\right] = w^\ddagger(v^\dagger, w^\dagger), \end{aligned} \quad (3.17)$$

where the identity $\langle \epsilon, v^\dagger - w^\dagger\rangle = \alpha^+$ (from Lemma 3.1) and the property of the reduced masses $\mu_{12} = \mu_{34}$ were used in (3.16)–(3.17).

Similarly, since $\mu_{12} = \mu_{34}$, one observes that

$$(\langle\epsilon, v-w\rangle)^2 \geq 2(\gamma_1 - E_{abs})/\mu_{34} \iff \left(\langle\epsilon, v^\dagger - w^\dagger\rangle\right)^2 = (\langle\epsilon, v-w\rangle)^2 + \frac{2E_{abs}}{\mu_{34}} \geq \frac{2(\gamma_1 - E_{abs})}{\mu_{34}} + \frac{2E_{abs}}{\mu_{34}} = \frac{2\gamma_1}{\mu_{12}}, \tag{3.18}$$

thus implying that $\Theta(\langle\epsilon, v-w\rangle - \Gamma_{34})$ in (3.14′)–(3.15′) can by replaced by $\Theta(\langle\epsilon, v^\dagger - w^\dagger\rangle - \Gamma_{12})$.

Now, combining (3.16)–(3.17) and (3.18), (3.14′)–(3.15′) take the form

$$\int\limits_{\mathbb{R}^3} \phi_3 J_3^R \, dv = \beta_{34}\sigma_{34}^2 \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \phi_3(v^\ddagger)\left[f_1(v^\dagger)f_2(w^\dagger) - f_3(v^\ddagger)f_4(w^\ddagger)\right]\langle\epsilon, v^\dagger - w^\dagger\rangle\Theta(\langle\epsilon, v^\dagger - w^\dagger\rangle - \Gamma_{12})\, d\epsilon dw^\dagger dv^\dagger \tag{3.14′′}$$

and

$$\int\limits_{\mathbb{R}^3} \phi_4 J_4^R \, dv = \beta_{43}\sigma_{43}^2 \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \phi_4^\ddagger(v)\left[f_2(v^\dagger)f_1(w^\dagger) - f_4(v^\ddagger)f_3(w^\ddagger)\right]\langle\epsilon, v^\dagger - w^\dagger\rangle\Theta(\langle\epsilon, v^\dagger - w^\dagger\rangle - \Gamma_{12})\, d\epsilon dw^\dagger dv^\dagger. \tag{3.15′′}$$

Next, change of the variables $(v, w, \epsilon) \mapsto (w, v, -\epsilon)$ in (3.13) and (3.15′′) together with renaming the integration variables from $(v^\dagger, w^\dagger)$ to (v, w) in (3.14′′)–(3.15′′), and finally summing up the resulting left hand sides of (3.12)–(3.15), results in (3.5).

Proof of (3.6) follows the same line of arguments; this time however, one changes the integration variables in (3.12)–(3.13) from (v, w) to $(v^\ddagger, w^\ddagger)$. In this process v and w, as the functions of $v^\ddagger$, $w^\ddagger$, become $v^\dagger$ and $w^\dagger$, respectively. □

Remark 1. The assumption in Proposition 3.1 that $f_i \in C_0(\Omega \times \mathbb{R}^3)$, for $i = 1, \ldots, 4$, is only needed to make sure that all the integrals exist and are finite.

Proposition 3.1 has been proven under the conditions that $\beta_{ij} = \beta_{ji}$ for $(i, j) \in \{(1, 2), (2, 1), (3, 4), (4, 3)\}$, and $\mu_{12} = \mu_{34}$. Although it is possible to obtain extensions of Proposition 3.1 without any assumptions on β_{ij}, I will not consider here these generalizations. Furthermore, since we already have the identities $\sigma_{12} = \sigma_{34} = \sigma_{21} = \sigma_{43}$, in order to have the conservation laws (of mass, momentum, and energy) built into the model one has to require that $\beta_{12} = \beta_{34}$. Below, I state the conditions that will be assumed from now on in this work:

Condition 1.

(1) Reactive distances: $\sigma_{12} = \sigma_{34}$,

(2) Masses: $m_1 = m_3$ and $m_2 = m_4$, (implying $\mu_{12} = \mu_{34}$)

(3) Steric factors: $0 \leq \beta_{12} \leq 1$ and $\beta_{12} = \beta_{21} = \beta_{34} = \beta_{43}$,

(4) Internal energies: $E_i > 0$, $i = 1, \ldots, 4$, and $E_{abs} = E_3 + E_4 - E_2 - E_1 > 0$,

(5) Activation energies: $\gamma_1 = \gamma_2 \geq E_{abs}$ and $\gamma_3 = \gamma_4 = \gamma_1 - E_{abs}$.

Now, under Condition 1 and in view of (3.4) and (3.5), one has, for any $a, c \in \mathbb{R}$ and $b \in \mathbb{R}^3$,

$$\phi_i(x,v) = am_i + m_i\langle b, v\rangle + c\left(\frac{m_i v^2}{2} + E_i\right), \ i = 1, \ldots, 4, \quad \Longrightarrow \quad \begin{cases} \sum\limits_{i=1}^{4} \int\limits_{\mathbb{R}^3} \phi_i J_i^E \, dv = 0, \\ \sum\limits_{i=1}^{4} \int\limits_{\mathbb{R}^3} \phi_i J_i^R \, dv = 0. \end{cases} \tag{3.19}$$

Property (3.19) implies that if f_i is a nonnegative smooth solution of (3.1) on $[0, T]$, $T > 0$, then, at least formally, we have the following conservation laws for $t \in [0, T]$:

$$\sum_{i=1}^{4} \iint\limits_{\Omega\times\mathbb{R}^3} m_i f_i(t,x,v)\, dvdx = \sum_{i=1}^{4} \iint\limits_{\Omega\times\mathbb{R}^3} m_i f_{i0}(x,v)\, dvdx, \qquad \text{(conservation of mass)} \tag{3.20}$$

$$\sum_{i=1}^{4} \iint\limits_{\Omega\times\mathbb{R}^3} m_i v f_i(t,x,v)\, dvdx = \sum_{i=1}^{4} \iint\limits_{\Omega\times\mathbb{R}^3} m_i v f_{i0}(x,v)\, dvdx, \qquad \text{(conservation of momentum)} \tag{3.21}$$

$$\sum_{i=1}^{4} \iint\limits_{\Omega\times\mathbb{R}^3} \left(\frac{m_i v^2}{2} + E_i\right) f_i(t,x,v)\, dvdx = \sum_{i=1}^{4} \iint\limits_{\Omega\times\mathbb{R}^3} \left(\frac{m_i v^2}{2} + E_i\right) f_{i0}(x,v)\, dvdx, \quad \text{(conservation of energy)} \tag{3.22}$$

where $f_{i0}(x,v)$, $i = 1, \ldots, 4$, are nonnegative initial conditions of the dilute SRS kinetic system (3.1). The above conservation laws follow easily from multiplying the dilute SRS system by corresponding ϕ_i, integrating with respect to $(t,x,v) \in [0,T] \times \Omega \times \mathbb{R}^3$, and using (3.19).

An additional conservation law (along the characteristics of the streaming operator in the left hand side of (3.1)) can be obtained by noticing that $\sum\limits_{i=1}^{4} \int\limits_{\mathbb{R}^3} \phi_i J_i^E \, dv = 0$ and $\sum\limits_{i=1}^{4} \int\limits_{\mathbb{R}^3} \phi_i J_i^R \, dv = 0$ also for $\phi_i(x,v) = m_i \dfrac{(x-tv)^2}{2} + E_i$ and any $t \in [0,T]$. Next, after multiplying dilute SRS kinetic system (3.1) by $m_i \dfrac{(x-tv)^2}{2} + E_i$

and integrating by parts, one has, for $t \in [0,T]$,

$$\sum_{i=1}^{4} \iint_{\Omega\times\mathbb{R}^3} \left(\frac{m_i\,(x-tv)^2}{2} + E_i\right) f_i(t,x,v)\,dvdx = \sum_{i=1}^{4} \iint_{\Omega\times\mathbb{R}^3} \left(\frac{m_i x^2}{2} + E_i\right) f_{i0}(x,v)\,dvdx, \tag{3.23}$$

Similarly to the cases of the kinetic equations for a single specie (see, for example, [8], [14]), the conservation laws (3.20)–(3.23) and non-negativity of f_i, f_{i0} yield the following estimation

$$\sup_i \sup_{t\in[0,T]} \iint_{\Omega\times\mathbb{R}^3} x^2 f_i(t,x,v)\,dvdx \le C_1, \tag{3.24}$$

where $C_1 > 0$ depends only on T, $\sup_i \iint_{\Omega\times\mathbb{R}^3} x^2 f_i\,dvdx$, and on $\sup_i \iint_{\Omega\times\mathbb{R}^3} (1+v^2) f_i\,dvdx$.

Remark 2. The estimation (3.24) is superfluous in the case $\Omega = [0,L]^3$.

Proposition 3.1 also implies existence of a Liapunov functional (an H-function) for (3.1), consistent with system's physical equilibrium. For f_i a smooth nonnegative solution, we multiply (3.1) by $1+\log f_i$, integrate over $\Omega\times\mathbb{R}^3$, and use (3.4)–(3.5) (with $\phi_i = \log f_i$) to obtain the following entropy identity:

$$\begin{aligned}
&\frac{d}{dt}\sum_{i=1}^{4} \iint_{\Omega\times\mathbb{R}^3} f_i \log f_i\,dvdx \\
&+ \sum_{i,s=1}^{4} \sigma_{is}^2 \int\!\!\cdots\!\!\int_{\Omega\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \Big[f_i(v')f_s(w') - f_i(v)f_s(w)\Big] \log\left(\frac{f_i(v')f_s(w')}{f_i(v)f_s(w)}\right)\Theta(\langle\epsilon, v-w\rangle)\langle\epsilon, v-w\rangle \Xi_{is}\,d\epsilon dwdvdx \\
&+ \beta_{12}\sigma_{12}^2 \int\!\!\cdots\!\!\int_{\Omega\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \Big\{\Big[f_3(v^{\ddagger})f_4(w^{\ddagger}) - f_1(v)f_2(w)\Big]\times \\
&\log\left(\frac{f_3(v^{\ddagger})f_4(w^{\ddagger})}{f_1(v)f_2(w)}\right)\Theta(\langle\epsilon, v-w\rangle - \Gamma_{12})\langle\epsilon, v-w\rangle\Big\}\,d\epsilon dwdvdx = 0,
\end{aligned} \tag{3.25}$$

with Ξ_{is} given in (3.7). It is important to notice that the second and the third terms in the left hand side of (3.25) are nonnegative. Indeed, this follows from the inequalities

$$\Big[f_i(v')f_s(w') - f_i(v)f_s(w)\Big]\log\left(\frac{f_i(v')f_s(w')}{f_i(v)f_s(w)}\right) \ge 0, \quad \Big[f_3(v^{\ddagger})f_4(w^{\ddagger}) - f_1(v)f_2(w)\Big]\log\left(\frac{f_3(v^{\ddagger})f_4(w^{\ddagger})}{f_1(v)f_2(w)}\right) \ge 0, \tag{3.26}$$

for any $i, s = 1, \dots, 4$. Next, integrating (3.25) over $0 \le t_1 \le \tau \le t_2 \le T$ and using (3.26), one obtains the corresponding H-theorem,

$$\sum_{i=1}^{4} \iint_{\Omega\times\mathbb{R}^3} f_i(t_2)\log f_i(t_2)\, dvdx \le \tag{3.27}$$

$$\sum_{i=1}^{4} \iint_{\Omega\times\mathbb{R}^3} f_i(t_2)\log f_i(t_2)\, dvdx + \int_{t_1}^{t_2} \iint_{\Omega\times\mathbb{R}^3} \Delta(v,\{f_i\})\, dvdx \equiv$$

$$\sum_{i=1}^{4} \iint_{\Omega\times\mathbb{R}^3} f_i(t_2)\log f_i(t_2)\, dvdx +$$

$$\sum_{i,s=1}^{4} \sigma_{is}^2 \int_{t_1}^{t_2} \int\cdots\int_{\Omega\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \Big[f_i(v')f_s(w') - f_i(v)f_s(w)\Big] \log\left(\frac{f_i(v')f_s(w')}{f_i(v)f_s(w)}\right)\Theta(\langle\epsilon, v-w\rangle)\langle\epsilon, v-w\rangle \Xi_{is}\, d\epsilon dw dv dx d\tau$$

$$+\beta_{12}\sigma_{12}^2 \int_{t_1}^{t_2} \int\cdots\int_{\Omega\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \Big\{\Big[f_3(v^\ddagger)f_4(w^\ddagger) - f_1(v)f_2(w)\Big]\times$$

$$\log\left(\frac{f_3(v^\ddagger)f_4(w^\ddagger)}{f_1(v)f_2(w)}\right)\Theta(\langle\epsilon, v-w\rangle - \Gamma_{12})\langle\epsilon, v-w\rangle\Big\}\, d\epsilon dw dv dx d\tau$$

$$=\sum_{i=1}^{4} \iint_{\Omega\times\mathbb{R}^3} f_i(t_1)\log f_i(t_1)\, dvdx,$$

since $\Delta(v,\{f_i\}) \ge 0$. This shows that, for a nonnegative solution f_i of (3.1), the convex function $H(t)$ defined by

$$H(t) = \sum_{i=1}^{4} \iint_{\Omega\times\mathbb{R}^3} f_i(t,x,v)\log f_i(t,x,v)\, dvdx \tag{3.28}$$

is non-increasing in t.

Remark 3. If instead of (3.5) one uses identity (3.6), then the following (equivalent) entropy identities is true:

$$\frac{d}{dt}\sum_{i=1}^{4} \iint_{\Omega\times\mathbb{R}^3} f_i \log f_i\, dvdx$$

$$+\sum_{i,s=1}^{4} \sigma_{is}^2 \int\cdots\int_{\Omega\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \Big[f_i(v')f_s(w') - f_i(v)f_s(w)\Big] \log\left(\frac{f_i(v')f_s(w')}{f_i(v)f_s(w)}\right)\Theta(\langle\epsilon, v-w\rangle)\langle\epsilon, v-w\rangle \Xi_{is}\, d\epsilon dw dv dx$$

$$+\beta_{12}\sigma_{12}^2 \int\cdots\int_{\Omega\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \Big\{\Big[f_1(v^\dagger)f_2(w^\dagger) - f_3(v)f_4(w)\Big]\times$$

$$\log\left(\frac{f_1(v^\dagger)f_2(w^\dagger)}{f_3(v)f_4(w)}\right)\Theta(\langle\epsilon, v-w\rangle - \Gamma_{34})\langle\epsilon, v-w\rangle\Big\}\, d\epsilon dw dv dx = 0. \tag{3.29}$$

With the help of the entropy identity (3.25) and the inequalities (3.26) one can describe the equilibria solutions of (3.1). First, it is convenient to define macroscopic quantities (as the moments of f_i): the number densities

$n(t,x)$, the macroscopic velocity $u(t,x)$, and the macroscopic temperature $\mathcal{T}(t,x)$:

$$n_i(t,x) = \int\limits_{\mathbb{R}^3} f_i(t,x,v)\,dv, \qquad n(t,x) = \sum_{i=1}^{4} n_i(t,x); \tag{3.30}$$

$$u(t,x) = \frac{\sum\limits_{i=1}^{4} m_i n_i(t,x) u_i(t,x)}{\sum\limits_{i=1}^{4} m_i n_i(t,x)}, \qquad u_i(t,x) = \frac{\int\limits_{\mathbb{R}^3} v f_i(t,x,v)\,dv}{n_i(t,x)}; \tag{3.31}$$

$$3kn(t,x)\mathcal{T}(t,x) = \sum_{i=1}^{4} m_i \int\limits_{\mathbb{R}^3} [v - u(t,x)]^2 f_i(t,x,v)\,dv; \tag{3.32}$$

where k is the Boltzmann constant.

Proposition 3.2 (**Equilibrium solutions**). *Assume Condition 1 and let* $n_i(t,x) \geq 0$, $u(t,x)$, *and* $\mathcal{T}(t,x) \geq 0$ *be given measurable functions. Then for all* $0 \leq f_i \in L^1(\Omega \times \mathbb{R}^3)$ *with the moments given by* (3.30)–(3.32) *the following statements are equivalent:*

(1) $f_i = n_i \left(\frac{m_i}{2\pi k\mathcal{T}}\right)^{3/2} \exp\left(-\frac{m_i(v-u)^2}{2k\mathcal{T}}\right)$, $i = 1,\ldots,4$, *and* $n_1 n_2 = n_3 n_4 \exp\left(\frac{E_{abs}}{k\mathcal{T}}\right)$,

(2) $J_i^E(\{f_i\}) = 0$ *and* $J_i^R(\{f_i\}) = 0$, $i = 1,\ldots,4$,

(3) $\sum\limits_{i=1}^{4} \int\limits_{\mathbb{R}^3} \left[J_i^E(\{f_i\}) + J_i^R(\{f_i\})\right] \log f_i\,dv = 0.$

Proof. I proceed by showing that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). The proof of the first implication follows from substituting f_i, given in (1), into the collision integrals J_i^E and J_I^R and applying the conservation of mass, momentum, and energy on the microscopical level (see (2.6) and (2.7) for the corresponding identities). The second implication (i.e., (2) $\Rightarrow$ (3)) is trivially satisfied. In order to show the last implication (3) $\Rightarrow$ (1) one observes that using Proposition 3.1

$$\begin{aligned}
0 &= \sum_{i=1}^{4} \int\limits_{\mathbb{R}^3} \left[J_i^E(\{f_i\}) + J_i^R(\{f_i\})\right] \log f_i\,dv \\
&= \sum_{i,s=1}^{4} \sigma_{is}^2 \int\limits_{\Omega\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \!\!\cdots\!\! \int \left[f_i(v')f_s(w') - f_i(v)f_s(w)\right] \log\left(\frac{f_i(v')f_s(w')}{f_i(v)f_s(w)}\right) \Theta(\langle\epsilon, v-w\rangle)\langle\epsilon, v-w\rangle \Xi_{is}\,d\epsilon dw dv dx \\
&+ \beta_{12}\sigma_{12}^2 \int\limits_{\Omega\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \!\!\cdots\!\! \int \Bigg\{ \left[f_3(v^\ddagger)f_4(w^\ddagger) - f_1(v)f_2(w)\right] \times \\
&\log\left(\frac{f_3(v^\ddagger)f_4(w^\ddagger)}{f_1(v)f_2(w)}\right) \Theta(\langle\epsilon, v-w\rangle - \Gamma_{12})\langle\epsilon, v-w\rangle \Bigg\}\,d\epsilon dw dv dx,
\end{aligned} \tag{3.33}$$

where, as before, I suppressed t and x dependence in f_i. Next, inequalities $\Xi_{is} \geq 0$ and (3.26) together with fact that the for the function $f(y,z) = (y-z)\log\left(\frac{y}{z}\right) \geq 0,\ y,z>0$, the equality sign, $(y-z)\log\left(\frac{y}{z}\right) = 0$, holds if and only if $y=z$, yield the set of functional identities for f_i, $i=1,\dots,4$,

$$f_i(v')f_s(w') = f_i(v)f_s(w), \quad \text{almost everywhere in } (v,w) \in \mathbb{R}^3 \times \mathbb{R}^3, \quad i,s = 1,\dots 4, \tag{3.34}$$

$$f_3(v^\ddagger)f_4(w^\ddagger) = f_1(v)f_2(w), \quad \text{almost everywhere in } (v,w) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{3.35}$$

The solution to (3.34) is well known from the kinetic theory of non-reactive mixtures (see, for example, [13]):

$$f_i(v) = \exp\left(a_i + \langle b_i, v\rangle + c_i v^2\right), \tag{3.36}$$

for some $a_i,\ c_i \in \mathbb{R}$ and $b_i \in \mathbb{R}^3$. Next, integrability conditions together with the normalization and constraints (3.30)–(3.32) imposed on f_i imply that

$$f_i = n_i \left(\frac{m_i}{2\pi k\mathcal{T}}\right)^{3/2} \exp\left(-\frac{m_i(v-u)^2}{2k\mathcal{T}}\right), \quad i=1,\dots,4. \tag{3.37}$$

Finally, for f_i given in (3.37), the identity (3.35) is easily seen to be equivalent to $n_1 n_2 = n_3 n_4 \exp\left(\frac{E_{abs}}{k\mathcal{T}}\right)$, which is expresses equilibrium reaction rate of the chemical processes in the mixture. □

Remark 4. For the proof of equilibrium solutions, found in Proposition 3.2, I utilized identities (3.4)–(3.5) of Proposition 3.1. If instead of (3.5) one uses identity (3.6), then, in the proof, identity (3.35) is replaced by

$$f_1(v^\dagger)f_2(w^\dagger) = f_3(v)f_4(w), \quad \text{almost everywhere in} \quad (v,w) \in \mathbb{R}^3 \times \mathbb{R}^3, \tag{3.38}$$

which, for f_i in (3.37), is also equivalent to $n_1 n_2 = n_3 n_4 \exp\left(\frac{E_{abs}}{k\mathcal{T}}\right)$.

4. Existence results for the dilute SRS system

The proof of the existence theorem presented bellow follows the ideas developed by R. DiPerna and P. L. Lions ([8]) for the non-reactive single specie Boltzmann equation. It has three ingredients: (1) use of the L^1-weak compactness argument that follows from the conservation laws (3.20)–(3.22) and the entropy identity (3.25), (2) the velocity averaging lemma ([15]), and (3) a suitable notion of mild (or renormalized) solutions.

For the weak compactness argument one notices that if a nonnegative initial value f_{i0} of the evolution system (3.1) satisfies

$$\sup_i \iint_{\Omega\times\mathbb{R}^3} \left(1 + x^2 + v^2 + \log^+ f_{i0}\right) f_{i0}\, dv dx = C_0 < \infty, \tag{4.1}$$

then (3.20)–(3.22), (3.24), and the entropy identity (3.25) yield the following estimation for a smooth and nonnegative solution, f_i, of (3.1)

$$\sup_i \sup_{0\le t\le T} \iint\limits_{\Omega\times\mathbb{R}^3} \left(1+x^2+v^2+\log^+ f_i\right) f_i \, dvdx = C_T < \infty, \tag{4.2}$$

where $\log^{\pm}(z) = \max\{\pm\log(z), 0\}$.

Remark 5. When $\Omega = [0, L]^3$, constant C_T in (4.2) is independent of T.

Estimation (4.2) (the Dunford-Pettis theorem, see, for example, [16]) implies that the family of solutions $\{f_i(t) : 0 \le t \le T\}$ is relatively weakly compact in $L^1(\Omega \times \mathbb{R}^3)$.

From the physical point of view, (4.1) means that we start with the system that has finite total mass, momentum, energy, as well as finite initial entropy. In fact, at least at equilibrium and in the non-reactive situations, $-H(t)$, where $H(t)$ is an H-function defined in (3.28), represents an entropy of the considered system. Furthermore, (4.2) shows that these properties are maintained during the evolution of the system.

For the proof of estimation (4.2) it is enough to notice that (3.27), with $t_2 = t$ and $t_1 = 0$, yields (since $\Delta(v, \{f_i\}) \ge 0$) for $0 \le t \le T$,

$$\sup_i \iint\limits_{\Omega\times\mathbb{R}^3} f_i(t)\log^+ f_i(t)\, dvdx \le \sum_{i=1}^4 \iint\limits_{\Omega\times\mathbb{R}^3} f_i(t)\log^+ f_i(t)\, dvdx + \int_0^t \iint\limits_{\Omega\times\mathbb{R}^3} \Delta(v,\{f_i\})\, dvdx \tag{4.3}$$

$$= \sum_{i=1}^4 \iint\limits_{\Omega\times\mathbb{R}^3} f_i(t)\log^- f_i(t)\, dvdx + \sum_{i=1}^4 \iint\limits_{\Omega\times\mathbb{R}^3} f_{i0}\log^+ f_{i0}\, dvdx - \sum_{i=1}^4 \iint\limits_{\Omega\times\mathbb{R}^3} f_{i0}\log^- f_{i0}\, dvdx. \tag{4.4}$$

Next, use of the inequality $z\log(z/y) \ge -y$ with $y = \exp(-x^2 - v^2)$ and $z = f_i$ together with estimation (4.1) and boundedness of $\sup_i \sup_{0\le t\le T} \int_{\Omega\times\mathbb{R}^3} (1 + x^2 + v^2) f_i \, dvdx$ implies

$$\sup_i \sup_{0\le t\le T} \iint\limits_{\Omega\times\mathbb{R}^3} f_i(t)\log^+ f_i(t)\, dvdx \le \overline{C}_T, \tag{4.5}$$

and ultimately, (4.2).

Estimation (4.5) has another important physical interpretation: there can be now concentration of densities in the system. Indeed, using the Dunford-Pettis theorem (see, for example, [16]) one obtains that the family of macroscopic densities $\{n_i(t, x) : 0 \le t \le T\}$ is uniformly integrable, i.e., to each $\lambda > 0$ there corresponds a $\delta > 0$ such that

$$\sup_i \sup_{0\le t\le T} \int\limits_E n_i(t, x)\, dx < \lambda \tag{4.6}$$

for any $E \subset \Omega$ with $\mathrm{vol}(E) < \delta$. In particular, when $\Omega = [0, L]^3$, T can be set to ∞ in (4.6).

The next step consists in finding suitable approximations J_{in}^E and J_{in}^R of J_i^E and J_i^R, respectively, for which the problem

$$\frac{\partial f_i^n}{\partial t} + v\frac{\partial f_i^n}{\partial x} = J_{in}^E + J_{in}^R, \quad f_i^n(0, x, v) = f_{i0}^n(x, v), \quad i = 1, \dots, 4, \quad (x, v) \in \Omega \times \mathbb{R}^3 \tag{4.7}$$

can be solved by known methods. Then, one takes the weak limit $f_i^n \xrightarrow[n\to\infty]{} f_i$, and tries to show that f_i satisfies (3.1) in some specified sense. An important criterion of *suitable* approximations for J_i^E and J_i^R is that J_{in}^E and J_{in}^R must satisfy the properties listed in Proposition 3.1. These properties alone yield the crucial weak compactness estimation (4.2). The collision integrals J_i^E and J_i^R are not weakly continuous in $L^1(\Omega \times \mathbb{R}^3)$ (in fact, they are even difficult to define in a reasonable way in $L^1(\Omega \times \mathbb{R}^3)$), thus the passage to the limit in (4.7) cannot be achieved without additional tools. This brings us to the remaining two ingredients of DiPerna-Lions method. The velocity averaging lemma provides an additional compactness argument needed in the passage to the limit in (4.7).

Lemma 4.1 (**Velocity averaging** ([15])). *If $h_n \in L^1((0, T) \times \Omega \times \mathbb{R}^3)$ and $g_n \in L^1_{loc}((0, T) \times \Omega \times \mathbb{R}^3)$ satisfy the following transport equation*

$$T_v h_n \equiv \frac{\partial h_n}{\partial t} + v\frac{\partial h_n}{\partial x} = g_n, \tag{4.8}$$

in $\mathcal{D}'((0, T) \times \Omega \times \mathbb{R}^3)$, and for each compact set $K \subset (0, T) \times \Omega \times \mathbb{R}^3$ the sequences $\{h_n\}$ and $\{g_n\}$ are relatively weakly compact in $L^1((0, T) \times \Omega \times \mathbb{R}^3)$ and $L^1(K)$, respectively, then for all $\phi \in L^\infty((0, T) \times \Omega \times \mathbb{R}^3)$ the set

$$\left\{ \int\limits_{\mathbb{R}^3} \phi(t, x.v) f_n(t, x, v)\, dv : \ n = 1, 2, \dots \right\} = \left\{ \int\limits_{\mathbb{R}^3} \phi(t, x.v)(T_v^{-1} g_n)(t, x, v)\, dv : \ n = 1, 2, \dots \right\}, \tag{4.9}$$

is relatively compact in $L^1((0, T) \times \Omega)$.

Velocity averaging compensates for lack of regularity of T_v on the set of characteristic directions.

Definition 4.1. A nonnegative $f_i \in L^1_{loc}((0, T) \times \Omega \times \mathbb{R}^3)$ is a renormalized solution of (3.1) if

$$\frac{1}{1 + f_i} J_i^{E\pm} \in L^1_{loc}((0, T) \times \Omega \times \mathbb{R}^3), \quad \frac{1}{1 + f_i} J_i^{R\pm} \in L^1_{loc}((0, T) \times \Omega \times \mathbb{R}^3), \tag{4.10}$$

and

$$\frac{\partial}{\partial t} \log(1 + f_i) + v\frac{\partial}{\partial x} \log(1 + f_i) = \frac{1}{1 + f_i} \left[J_i^E + J_i^R \right], \tag{4.11}$$

in $\mathcal{D}'((0,T)\times\Omega\times\mathbb{R}^3)$, where

$$J_i^E = J_i^{E+} - J_i^{E-}, \quad J_i^R = J_i^{R+} - J_i^{R-}, \tag{4.12}$$

with $J_i^{E\pm}$ and $J_i^{R\pm}$ given by

$$\begin{aligned} J_i^{E+} = \sum_{s=1}^{4}\Bigg\{\sigma_{is}^2 \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} f_i(t,x,v')f_s(t,x,w')\Theta\left(\langle\epsilon, v-w\rangle\right)\langle\epsilon, v-w\rangle\, d\epsilon dw\Bigg\} \\ - \beta_{ij}\sigma_{ij}^2 \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} f_i(t,x,v')f_s(t,x,w')\Theta\left(\langle\epsilon, v-w\rangle - \Gamma_{ij}\right)\langle\epsilon, v-w\rangle\, d\epsilon dw, \end{aligned} \tag{4.13}$$

$$\begin{aligned} J_i^{E-} = f_i(t,x,v)\sum_{s=1}^{4}\Bigg\{\sigma_{is}^2 \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} f_s(t,x,w)\Theta\left(\langle\epsilon, v-w\rangle\right)\langle\epsilon, v-w\rangle\, d\epsilon dw\Bigg\} \\ - \beta_{ij}\sigma_{ij}^2 f_i(t,x,v) \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} f_s(t,x,w)\Theta\left(\langle\epsilon, v-w\rangle - \Gamma_{ij}\right)\langle\epsilon, v-w\rangle\, d\epsilon dw, \end{aligned} \tag{4.14}$$

and

$$J_i^{R+} = \beta_{ij}\sigma_{ij}^2 \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} f_k(t,x,v_{ij}^{\odot})f_l(t,x,w_{ij}^{\odot})\Theta\left(\langle\epsilon, v-w\rangle - \Gamma_{ij}\right)\langle\epsilon, v-w\rangle\, d\epsilon dw, \tag{4.15}$$

$$J_i^{R-} = \beta_{ij}\sigma_{ij}^2 f_i(t,x,v) \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} f_j(t,x,w)\Theta\left(\langle\epsilon, v-w\rangle - \Gamma_{ij}\right)\langle\epsilon, v-w\rangle\, d\epsilon dw, \tag{4.16}$$

respectively.

In (4.15)–(4.16), as before, $(v_{ij}^{\odot}, w_{ij}^{\odot}) = (v^{\ddagger}, w^{\ddagger})$ for $i,j=1,2$, and $(v_{ij}^{\odot}, w_{ij}^{\odot}) = (v^{\dagger}, w^{\dagger})$ for $i,j=3,4$. Also the index pairs (i,j) and (k,l) are associated with the set of indices (i,j,k,l) specified in (2.9).

Remark 6. The steric factors $0 \le \beta_{ij} \le 1$, therefore both operators J_i^{E+} and J_i^{E-} (at least formally) map nonnegative functions into nonnegative functions.

Next, let us assume for a moment that one already has found *suitable* approximations J_{in}^E and J_{in}^R to (3.1). If for $i=1,2,3,4$, $\{f_i^n\}_{n=1}^{\infty}$ is a sequence of nonnegative, solutions to (4.7) satisfying (4.2), uniformly in n, then, for $=1,2,3,4$ and $\delta>0$, $f_i^{n\delta} = (1/\delta)\log(1+\delta f_i^n)$ satisfies $0 \le f_i^{n\delta} \le f_i^n$. Thus, the sequence $\{f_i^{n\delta}\}_{n=1}^{\infty}$ is relatively weakly compact in $L^1((0,T)\times\Omega\times\mathbb{R}^3)$ and satisfies

$$\frac{\partial}{\partial t} f_i^{n\delta} + v\frac{\partial}{\partial x} f_i^{n\delta} = \frac{1}{1+\delta f_i^n}\left[J_{in}^E(\{f_i^n\}) + J_{in}^R(\{f_i^n\})\right]. \tag{4.17}$$

The averaging velocity Lemma 4.1 yields that the sequence $\{\int_{\mathbb{R}^3} \phi f_i^{n\delta}\, dv\}_{n=1}^{\infty}$, for each fixed $i = 1, 2, 3, 4$ and $\delta > 0$, is relatively compact in $L^1((0,T)\times\Omega)$, for all $\phi \in L^\infty((0,T)\times\Omega\times\mathbb{R}^3)$, if the sequence

$$\left\{ \frac{1}{1+\delta f_i^n} \left[J_{in}^E(\{f_i^n\}) + J_{in}^R(\{f_i^n\}) \right] \right\}_{n=1}^{\infty} \quad \text{is relatively weakly compact in } L^1((0,T)\times\Omega\times B_r), \tag{4.18}$$

with $B_r = \{y \in \mathbb{R}^3 : |z| \le r\}$. For the proof of (4.18) one needs gain-loss comparison estimates (a similar estimation appears in the case of a single specie Boltzmann equation [8]). For simplicity, I formulate them only for the original collision integrals J_i^E and J_i^R.

Lemma 4.2. *For $i, s = 1, 2, 3, 4$ and any $M > 1$*

$$J_i^{E+}(\{f_i^n\}) \le M J_i^{E-}(\{f_i^n\}) \tag{4.19}$$

$$+ \sigma_{is}^2 \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} \Big[f_i(v')f_s(w') - f_i(v)f_s(w) \Big] \log\left(\frac{f_i(v')f_s(w')}{f_i(v)f_s(w)} \right) \Theta(\langle\epsilon, v-w\rangle)\langle\epsilon, v-w\rangle \overline{\Xi}_{is}\, d\epsilon dw$$

$$J_1^{R+}(f_3^n, f_4^n) \le M J_1^{E-}(f_1^n, f_2^n) \tag{4.20}$$

$$+ \frac{1}{\log M}\beta_{12}\sigma_{12}^2 \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} \Big[f_3^n(v^\ddagger)f_4^n(w^\ddagger) - f_1^n(v)f_2^n(w) \Big] \log\left(\frac{f_3^n(v^\ddagger)f_4^n(w^\ddagger)}{f_1^n(v)f_2^n(w)} \right) \Theta(\langle\epsilon, v-w\rangle - \Gamma_{12})\langle\epsilon, v-w\rangle\, d\epsilon dw,$$

$$J_2^{R+}(f_3^n, f_4^n) \le M J_2^{E-}(f_1^n, f_2^n) \tag{4.21}$$

$$+ \frac{1}{\log M}\beta_{12}\sigma_{12}^2 \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} \Big[f_4^n(v^\ddagger)f_3^n(w^\ddagger) - f_2^n(v)f_1^n(w) \Big] \log\left(\frac{f_4^n(v^\ddagger)f_3^n(w^\ddagger)}{f_2^n(v)f_1^n(w)} \right) \Theta(\langle\epsilon, v-w\rangle - \Gamma_{12})\langle\epsilon, v-w\rangle\, d\epsilon dw,$$

$$J_3^{R+}(f_1^n, f_2^n) \le M J_3^{E-}(f_3^n, f_4^n) \tag{4.22}$$

$$+ \frac{1}{\log M}\beta_{34}\sigma_{34}^2 \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} \Big[f_1^n(v^\dagger)f_2^n(w^\dagger) - f_3^n(v)f_4^n(w) \Big] \log\left(\frac{f_1^n(v^\dagger)f_2^n(w^\dagger)}{f_3^n(v)f_4^n(w)} \right) \Theta(\langle\epsilon, v-w\rangle - \Gamma_{34})\langle\epsilon, v-w\rangle\, d\epsilon dw$$

$$J_4^{R+}(f_1^n, f_2^n) \le M J_4^{E-}(f_3^n, f_4^n) \tag{4.23}$$

$$+ \frac{1}{\log M}\beta_{34}\sigma_{34}^2 \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2} \Big[f_2^n(v^\dagger)f_1^n(w^\dagger) - f_4^n(v)f_3^n(w) \Big] \log\left(\frac{f_2^n(v^\dagger)f_1^n(w^\dagger)}{f_4^n(v)f_3^n(w)} \right) \Theta(\langle\epsilon, v-w\rangle - \Gamma_{34})\langle\epsilon, v-w\rangle\, d\epsilon dw,$$

where $\overline{\Xi}_{is}$ is given by

$$\overline{\Xi}_{is} = \begin{cases} \Theta(\langle\epsilon, v-w\rangle - \Gamma_{is}) + (1-\beta_{is})\Theta(\Gamma_{is} - \langle\epsilon, v-w\rangle), & \textit{if } (i,s) \in \{(1,2),(2,1),(3,4),(4,3)\}; \\ \Theta(\langle\epsilon, v-w\rangle), & \textit{otherwise.} \end{cases} \tag{4.24}$$

Each of the inequalities (4.19)–(4.23) can be proven in a similar way as in the case of a single specie Boltzmann equation (see [8], page 336).

Now, since f_i^n satisfy (4.2), uniformly in n, the sequences,

$$\left\{\frac{1}{1+\delta f_i^n} J_i^{E-}(\{f_i^n\})\right\}_{n=1}^{\infty}, \left\{\frac{1}{1+\delta f_i^n} J_i^{R-}(\{f_i^n\})\right\}_{n=1}^{\infty} \subset L^\infty((0,T);L^1(\Omega\times B_r)) \tag{4.25}$$

are relatively weakly compact in $L^1((0,T)\times\Omega\times B_r)$, for any $r>0$. Proof of this is similar to the single specie case (see, for example, [8], pp. 353-354). Next, the second terms of the right hand sides in each of the inequalities (4.19)–(4.20) are nonnegative and bounded above by the nonnegative function $4\Delta(v,\{f_i^n\})$ that appears in the H-theorem (3.27). The entropy identity (3.25) together with (4.2) yields that the set $\{\Delta(v,\{f_i^n\})\}_{n=1}^\infty$ is bounded in $L^1((0,T)\times\Omega\times\mathbb{R}^3)$, and the comparison principle implies the weak compactness in $L^1((0,T)\times\Omega\times B_r)$ of the sequences

$$\left\{\frac{1}{1+\delta f_i^n} J_i^{E+}(\{f_i^n\})\right\}_{n=1}^{\infty}, \left\{\frac{1}{1+\delta f_1^n} J_1^{R+}(f_3^n,f_4^n)\right\}_{n=1}^{\infty}. \tag{4.26}$$

The second term of the right hand side of (4.21) is nonnegative and its L^1-norm, after performing the change of variables $(v,w,\epsilon)\mapsto(w,v,-\epsilon)$, is bounded above by $\sup_n\|\Delta(f_i^n)\|_{L^1((0,T)\times\Omega\times\mathbb{R}^3)}<\infty$. As before, this proves the weak compactness of the sequence $\{(1+\delta f_2^n)^{-1}J_2^{R+}(f_3^n,f_4^n)\}_{n=1}^\infty$ in $L^1((0,T)\times\Omega\times B_r)$. The steps in proving the weak compactness of the sequences $\{(1+\delta f_3^n)^{-1}J_3^{R+}(f_1^n,f_2^n)\}_{n=1}^\infty$ and $\{(1+\delta f_4^n)^{-1}J_4^{R+}(f_1^n.f_2^n)\}_{n=1}^\infty$ are identical to the previous ones, with one exception that this time one uses the form of the entropy identity given in (3.29). This ends the proof of (4.18), at least for the original collision integrals. Once a *suitable* approximation is defined, it will become clear how to use the just given proof to show (4.18).

Finally, we have

Lemma 4.3. *If for each $i=1,2,3,4$ the nonnegative sequence $\{f_i^n\}_{n=1}^\infty$ satisfies (4.2), uniformly in n, and for each $\delta>0$ the sequence $\{\int_{\mathbb{R}^3}\phi f_i^{n\delta}\,dv\}_{n=1}^\infty$, with $f_i^{n\delta}=(1/\delta)\log(1+\delta f_i^n)$, is relatively compact in $L^1((0,T)\times\Omega)$, for all $\phi\in L^1((0,T)\times\Omega\times\mathbb{R}^3)$, then the sequence $\{\int_{\mathbb{R}^3}\phi f_i^n\,dv\}_{n=1}^\infty$ is also relatively compact in $L^1((0,T)\times\Omega)$, for all $\phi\in L^1((0,T)\times\Omega\times\mathbb{R}^3)$.*

Proof. Estimation (4.2) implies that $\{f_i^n\}_{n=1}^\infty$, $i=1,2,3,4$, are weakly relatively compact in $L^1(\Omega\times\mathbb{R}^3)$, thus it is enough to show that for all $\phi\in L^1((0,T)\times\Omega\times\mathbb{R}^3)$ and after passing to a subsequence, if necessary,

$$\int_{\mathbb{R}^3}\phi f_i^n\,dv \xrightarrow[n\to\infty]{} \int_{\mathbb{R}^3}\phi f_i\,dv \quad \text{strongly in } L^1((0,T)\times\Omega), \quad i=1,2,3,4, \tag{4.27}$$

where f_i is a weak limit of $\{f_i^n\}_{n=1}^{\infty}$. I claim that (4.27) follows if it can be shown that

$$\sup_n \sup_{0\le t\le T} \|f_i^n - f_i^{n\delta}\|_{L^1(\Omega\times\mathbb{R}^3)} \xrightarrow[\delta\to 0^+]{} 0, \quad i=1,2,3,4. \tag{4.28}$$

Indeed, since the norm is lower weakly semi-continuous, one obtains from (4.28)

$$\sup_{0\le t\le T} \|f_i - f_i^{\delta}\|_{L^1(\Omega\times\mathbb{R}^3)} \le \sup_{0\le t\le T} \liminf_{n\to\infty} \|f_i^n - f_i^{n\delta}\|_{L^1(\Omega\times\mathbb{R}^3)} \xrightarrow[\delta\to 0^+]{} 0, \quad i=1,2,3,4, \tag{4.29}$$

and

$$\int_{\mathbb{R}^3} f_i^n \phi\, dv = \int_{\mathbb{R}^3} (f_i^n - f_i^{n\delta})\phi\, dv + \int_{\mathbb{R}^3} (f_i^{n\delta} - f_i^{\delta})\phi\, dv + \int_{\mathbb{R}^3} f_i^{\delta}\phi\, dv, \quad i=1,2,3,4, \tag{4.30}$$

where f_i^{δ} is the weak limit of $\{f_i^{n\delta}\}_{n=1}^{\infty}$ satisfying, by the assumption,

$$\int_{\mathbb{R}^3} \phi f_i^{n\delta}\, dv \xrightarrow[n\to\infty]{} \int_{\mathbb{R}^3} \phi f_i^{\delta}\, dv \quad \text{strongly in } L^1((0,T)\times\Omega), \quad i=1,2,3,4. \tag{4.31}$$

Thus, the application of (4.28)–(4.31) gives (4.27) for all $\phi \in L^1((0,T)\times\Omega\times\mathbb{R}^3)$. Next, in order to prove (4.28) we notice that for all $R>0$

$$0 \le s - \frac{1}{\delta}\log(1+\delta s) \le s\left[\left(1 - \frac{\log(1+\delta s)}{\delta s}\right)\chi_{\{s\le R\}}\right] + s\chi_{\{s\ge R\}}, \tag{4.32}$$

with χ_A the characteristic function of the set A and $[1-\log(1+\delta s)/(\delta s)]\,\chi_{\{s\le R\}} \xrightarrow[\delta\to 0^+]{} 0$ locally uniformly in R. Finally, the estimation (4.2) implies

$$\sup_n \sup_{0\le t\le T} \iint_{\Omega\times\mathbb{R}^3} f_i^n \chi_{\{s\ge R\}}\, dvdx \xrightarrow[R\to\infty]{} 0, \quad i=1,2,3,4, \tag{4.33}$$

thus completing the proof of (4.28) and the lemma itself. □

Remark 7. In fact, using again (4.2), the convergence in (4.27) holds for ϕ with $\left(1+|x|^k+|v|^k\right)^{-1}\phi \in L^\infty((0,T)\times\Omega\times\mathbb{R}^3)$ and $0\le k<2$.

The strong convergence in (4.27) is fundamental in proving that a sequence of smooth solutions converges weakly to a renormalized solution.

I consider now the approximate problem,

$$\frac{\partial f_i^n}{\partial t} + v\frac{\partial f_i^n}{\partial x} = J_{in}^E + J_{in}^R, \quad f_i^n(0,x,v) = f_{i0}^n(x,v), \quad i=1,\dots,4, \quad (x,v)\in\Omega\times\mathbb{R}^3, \tag{4.34}$$

where

$$\left(1+\frac{1}{n}\sum_{m=1}^{4}\int_{\mathbb{R}^3} f_m^n\,dv\right) J_{in}^E \tag{4.35}$$
$$= \sum_{s=1}^{4}\left\{\sigma_{is}^2 \iint_{\mathbb{R}^3\times\mathbb{S}^2}\left[f_i^n(t,x,v')f_s^n(t,x,w') - f_i^n(t,x,v)f_s^n(t,x,w)\right]\Theta\left(\langle\epsilon, v-w\rangle\right) B_n^E(v,w,\epsilon)\,d\epsilon dw\right\}$$
$$-\beta_{ij}\sigma_{ij}^2 \iint_{\mathbb{R}^3\times\mathbb{S}^2}\left[f_i^n(t,x,v')f_s^n(t,x,w') - f_i^n(t,x,v)f_s^n(t,x,w)\right]\Theta\left(\langle\epsilon, v-w\rangle - \Gamma_{ij}\right) B_n^E(v,w,\epsilon)\,d\epsilon dw,$$

and

$$\left(1+\frac{1}{n}\sum_{m=1}^{4}\int_{\mathbb{R}^3} f_m^n\,dv\right) J_{in}^R \tag{4.36}$$
$$= \beta_{ij}\sigma_{ij}^2 \iint_{\mathbb{R}^3\times\mathbb{S}^2}\left[f_k^n(t,x,v_{ij}^{\odot})f_l^n(t,x,w_{ij}^{\odot}) - f_i^n(t,x,v)f_j^n(t,x,w)\right]\Theta\left(\langle\epsilon, v-w\rangle - \Gamma_{ij}\right) B_n^{(ij)}(v,w,\epsilon)\,d\epsilon dw,$$

with

$$B_n^E(v,w,\epsilon) = \begin{cases} \langle\epsilon, v-w\rangle, & \text{if } v^2+w^2 \le n; \\ 0, & \text{otherwise,} \end{cases} \tag{4.37}$$

and

$$B_n^{(ij)}(v,w,\epsilon) = \begin{cases} B_n^E(v,w,\epsilon), & \text{if } (i,j)\in\{(1,2),(2,1)\}; \\ \dfrac{\langle\epsilon, v-w\rangle}{\sqrt{\left(\langle\epsilon, v-w\rangle\right)^2 + 2E_{abs}/\mu_{34}}} B_n^E(v^\dagger,w^\dagger,\epsilon), & \text{if } (i,j)\in\{(3,4),(4,3)\}. \end{cases} \tag{4.38}$$

As before, the pair of velocities $(v_i^{\odot}, v_j^{\odot})$ refers to post-reactive velocities described either in (2.3) or (2.5), i.e., $(v_{ij}^{\odot}, w_{ij}^{\odot}) = (v^\ddagger, w^\ddagger)$ for $i,j=1,2$, and $(v_{ij}^{\odot}, w_{ij}^{\odot}) = (v^\dagger, w^\dagger)$ for $i,j=3,4$. Also, the index pairs (i,j) and (k,l) appearing in (4.35)-(4.36) are associated with the set of indices (i,j,k,l) specified in (2.9).

The initial distributions f_{i0}^n are given by

$$f_{i0}^n = \max\left\{\min\{f_{i0}, n\}, \frac{\rho}{n}\exp(-x^2-v^2)\right\}, \tag{4.39}$$

where $f_{i0}\ge 0$ satisfy (4.1) and $\rho>0$ is sufficiently small.

Observe that for $n\ge 1$, the approximate scattering kernels B_n^E, $B_n^{(ij)} \in L^\infty(\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2)$; they are also symmetric with respect to the change of variables $(v,w,\epsilon)\mapsto(w,v,-\epsilon)$. In addition, Lemma 3.1 implies that they converge pointwise (and in $L^1_{loc}(\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2)$ to $\langle\epsilon, v-w\rangle$ as $n\to\infty$. Furthermore, $f_{i0}^n \to f_{i0}$ in $L^1(\Omega\times\mathbb{R}^3)$ and, for each $n\ge 1$, $f_{i0}^n \in L^\infty(\Omega\times\mathbb{R}^3)$, $i=1,2,3,4$.

Remark 8. The important property of the approximate collision integrals defined in (4.35)-(4.36) is that they possess properties listed in Proposition 3.1. Indeed, using Lemma 3.1, is easy to see that the corresponding identities (3.4)–(3.6) hold with the expressions $\langle \epsilon, v-w\rangle$ replaced by $B_n^E(v,w,\epsilon)$, $B_n^{(12)}(v,w,\epsilon)$, and $B_n^{(34)}(v,w,\epsilon)$, respectively.

Next, I set up problem (4.34) in the framework of a semilinear evolution equation in the Banach space $X = \prod_{i=1}^4 L^1(\Omega\times\mathbb{R}^3)$ with the norm $\|f\|_X = \sup_i \iint_{\Omega\times\mathbb{R}^3} |f|\, dvdx$. Consider the operator in X,

$$Af \equiv (v\nabla_x f_1, v\nabla_x f_2, v\nabla_x f_3, v\nabla_x f_4), \tag{4.40}$$

with $f=(f_1,f_2,f_3,f_4)$. Then, A generates a strongly continuous contraction semigroup $U(t)$ in X. Next, for $f_0 = (f_{10}^n, f_{20}^n, f_{30}^n, f_{40}^n)$, I rewrite (4.34) as a semilinear evolution equation on the closed set $D_M \subset X$, $M>0$,

$$D_M = \left\{ (f_1,f_2,f_3,f_4)\in X : f_i \ge 0,\ \sum_{i=1}^4 \iint_{\Omega\times\mathbb{R}^3} \left(\frac{m_i v^2}{2} + E_i\right) f_i\, dvdx \le M \right\}, \tag{4.41}$$

in the form

$$\frac{d}{dt} f^n(t) + Af^n = F_n(f^n), \quad f^n(0) = f_0, \quad 0\le t\le T, \tag{4.42}$$

where

$$F_n(f^n) = \left(J_{1n}^E(\{f_i^n\}) + J_{1n}^R(\{f_i^n\}),\, J_{2n}^E(\{f_i^n\}) + J_{2n}^R(\{f_i^n\}),\, J_{3n}^E(\{f_i^n\}) + J_{3n}^R(\{f_i^n\}),\, J_{4n}^E(\{f_i^n\}) + J_{4n}^R(\{f_i^n\})\right), \tag{4.43}$$

with $f^n = (f_1^n, f_2^n, f_3^n, f_4^n)$. A continuous function f from $[0,T]$ into $D_M\subset X$ is a weak solution of (4.42) if it satisfies

$$f(t) = U(t)f_0 + \int_0^t U(t-s)F(f(s)\, ds \tag{4.44}$$

for $t\in[0,T]$. In (4.44), the integral is the Riemann integral in X, where for clarity, I suppressed the subscript n from f, f_0, and F. Among many theorems that guarantee the existence of weak solutions to semilinear evolution equations (4.44), the one below is suitable for our case (see, for example, Theorem 2.1, pp. 335 of [17]).

Theorem 4.1. *Assume that:*

(1) $U(t): D_M \mapsto D_M$ *is a strongly continuous semigroup in* X *generated by* A*,*

(2) *$F : D_M \mapsto X$ and there exists $K > 0$ such that*

$$\|F(f) - F(g)\|_X \leq K\|f - g\|_X, \quad f,\, g \in D_M$$

(3) *For $f \in D_M$*

$$\liminf_{h \to 0^+} dist(f + hF(f); D_M) = 0,$$

where

$$dist(f; D_M) = \inf_{g \in D_M} \|f - g\|_X$$

is the distance function from f to D_M.

Then there exists a unique weak solution f on $[0, T]$, for any $T > 0$.

Condition (3) of Theorem 4.1, known often as the Nagumo boundary condition for the set D_M, guarantees invariance of D_M under the time evolution.

Next, I check that the conditions of Theorem 4.1 are satisfied for A in (4.40) and F in (4.43). The action of $U(t)$ on $f = (f_1, f_2, f_3, f_4)$ is given by

$$(U(t)f)(x, v) = (f_1(x - tv, v), f_2(x - tv, v), f_3(x - tv, v), f_4(x - tv, v)), \tag{4.45}$$

thus, (1) is satisfied. For (2) it is enough to notice that since $f \in D_M,\ f_i \geq 0,$ and the multiplication factor $(1 + \frac{1}{n}\sum_{i=1}^{4} \int_{\mathbb{R}^3} f_i^n\, dv)^{-1}$ appearing in front the approximate collision integrals (4.35)–(4.36) makes the operator F Lipschitz continuous with constant K dependent on n. Finally, by splitting $J_{in}^E = J_{in}^{E+} - J_{in}^{E-}$ and $J_{in}^R = J_{in}^{R+} - J_{in}^{R-}$ in an analogical way as for the original collision integrals (4.13)–(4.16), one notes that for $f \in D_M,\ f_i + hJ_{in}^E(\{f_i\}) \geq 0$, $i = 1, 2, 3, 4$, for small enough $h > 0$; therefore the analog of Proposition 3.1 for the approximate collision integrals (4.35)–(4.36) (see Remark 8) with $\phi_i = m_i v^2/2 + E_i$ yields the Nagumo boundary condition (3).

In the last step before stating the main existence result, I recall two additional (equivalent) notions of solutions used in the original work of DiPerna-Lions [8].

Definition 4.2. A nonnegative $f_i \in L^1_{loc}((0, T) \times \Omega \times \mathbb{R}^3)$, $i = 1, 2, 3, 4$, is a mild solution of (3.1) if for each $0 < T < \infty,\ J_i^{E\pm}(\{f_i\}),\ J_i^{R\pm}(\{f_i\}) \in L^1(0, T)$, a.e. (almost everywhere) in $(x, v) \in \Omega \times \mathbb{R}^3$ and satisfies

$$f_i^{\#}(t, x, v) - f_i^{\#}(s, x, v) = \int_s^t \left[J_i^E(\{f_i\})^{\#}(\tau, x, v) + J_i^R(\{f_i\})^{\#}(\tau, x, v)\right] d\tau, \qquad 0 < s < t \leq T, \tag{4.46}$$

where $f_i^{\#}(t,x,v) = f(t,x+tv,v)$ and similarly for $J_i^{E\#}$ and $J_i^{R\#}$.

Following [8], one can show that f_i is mild solution if and only if f_i is a renormalized solution (Definition 4.1).

Finally, let $\mathcal{F}_i^{\#}(t,x,v) = \int_0^t L_i(\{f_i\})^{\#}(\tau,x,v)\,d\tau$, where $f_i L_i(\{f_i\}) = J_i^{E-}(\{f_i\}) + J_i^{R-}(\{f_i\})$, with J_i^{E-} and J_i^{R-} given in (4.14) and (4.16), respectively. If for $i = 1,2,3,4,\ T > 0\ L_i(\{f_i\}) \in L^1_{loc}((0,T)\times\Omega\times\mathbb{R}^3)$, then f_i is a mild solution of (3.1) if and only if f_i satisfies

$$\begin{aligned} f_i^{\#}(t,x,v) - f_i^{\#}(s,x,v)\exp\left\{-\left[\mathcal{F}_i^{\#}(t) - \mathcal{F}_i^{\#}(s)\right]\right\} \\ = \int_s^t \left[J_i^{E+}(\{f_i\}) + J_i^{R+}(\{f_i\})\right]^{\#}(\tau,x,v)\exp\left\{-\left[\mathcal{F}_i^{\#}(t) - \mathcal{F}_i^{\#}(\tau)\right]\right\}d\tau, \end{aligned} \tag{4.47}$$

for any $0 < s < t \le T$ and a.e. in $(x,v) \in \Omega\times\mathbb{R}^3$, $i = 1,2,3,4$. Here, J_i^{E+} and J_i^{R+} are given in (4.13) and (4.15), respectively.

Theorem 4.2 (Global existence result). *If for $i = 1,2,3,4$, $f_{i0} \ge 0$ satisfies condition* (4.1) *then there exists a nonnegative mild solution f_i of* (3.1)*, with $f_i \in C([0,T];L^1(\Omega\times\mathbb{R}^3))$ satisfying* (4.3)*, and such that $f_i(t)\big|_{t=0} = f_{i0}$, for $i = 1,2,3,4$.*

Proof. I will sketch the proof in several steps. For brevity, I will skip details of proofs that are very similar to the case of a single specie Boltzmann equation ([8]).

Step 1

From the identity $(f^n)^{\#}(t,x,v) = (U(-t)f^n)(t,x,v)$ it follows that the weak solutions f_i^n obtained in Theorem 4.1 are also mild solutions. Now, we observe that

$$\left\| \frac{1}{\left(1 + \frac{1}{n}\sum\limits_{m=1}^{4}\int\limits_{\mathbb{R}^3} f_m^n\,dv\right)} J_{in}^{E\pm}(\{f_i^n\}) \right\|_{L^\infty(\Omega\times\mathbb{R}^3)} \le C_n \sup_i \|f_i^n\|_{L^\infty(\Omega\times\mathbb{R}^3)}, \tag{4.48}$$

and

$$\left\| \frac{1}{\left(1 + \frac{1}{n}\sum\limits_{m=1}^{4}\int\limits_{\mathbb{R}^3} f_m^n\,dv\right)} J_{in}^{R\pm}(\{f_i^n\}) \right\|_{L^\infty(\Omega\times\mathbb{R}^3)} \le C_n \sup_i \|f_i^n\|_{L^\infty(\Omega\times\mathbb{R}^3)}. \tag{4.49}$$

The proof of the above estimates in the cases of J_{in}^{E+} and J_{in}^{R+} requires the change of integration from w to $V' = v' - w'$ and $V^{\ddagger} = v^{\ddagger} - w^{\ddagger}$ (or $V^{\dagger} = v^{\dagger} - w^{\dagger}$), respectively. Thus, since $f_{i0}^n \in L^\infty((0,T)\times\Omega\times\mathbb{R}^3)$, Gronwall's lemma applied to (4.44) gives L^∞-bound of approximate solutions. This bound depends on n and becomes arbitrary large as $n \to \infty$.

Now, combining $f_{i0}^n(x,v) \geq (\rho/n)\exp(-v^2-x^2)$, $J_{in}^{E-}(\{f_i^n\})+J_{in}^{R-}(\{f_i^n\}) = f_i^n L_{in}(\{f_i^n\})$, with L_{in} satisfying $\|L_{in}(\{f_i^n\})\|_{L^\infty(\Omega\times\mathbb{R}^3)} \leq \mathcal{C}_n$, one shows, similarly to the single specie case (see, [8], or [14]) that the approximate solution f_i^n satisfies, $i = 1,2,3,4$,

$$f_i^n(t,x,v) \geq \frac{\rho}{n}\exp(-\mathcal{C}_n t - |x-tv|^2 - v^2), \quad a.e \text{ in } (x,v) \in \Omega \times \mathbb{R}^3. \tag{4.50}$$

The bound (4.50) together with the absolute continuity of mild solutions f_i^n in t, a.e. in $(x,v) \in \Omega \times \mathbb{R}^3$, implies that $(f_i^n \log f_i^n)^\#(t,x,v)$ are absolutely continuous in t, a.e. in $(x,v) \in \Omega \times \mathbb{R}^3$, $i = 1,2,3,4$. This means that the indentity

$$\int_0^T \frac{d}{dt}\left(f_i^n \log f_i^n\right)^\#(t)\,dt = \left(f_i^n \log f_i^n\right)^\#(T) - f_{i0}^n \log f_{i0}^n, \quad \text{a.e. in } (x,v) \in \Omega \times \mathbb{R}^3. \tag{4.51}$$

is true. The mild solution f_i^n satisfies for $i = 1,2,3,4$ and a.e. in $(x,v) \in \Omega \times \mathbb{R}^3$

$$\frac{d}{dt}(f_i^n)^\# = J_{in}^E(\{f_i^n\})^\# - J_{in}^R(\{f_i^n\})^\#, \quad \text{a.e. in } t. \tag{4.52}$$

After multiplying (4.52) by $1 + (\log f_i^n)^\#$, summing over i and integrating over $\Omega \times \mathbb{R}^3$, and finally using (4.51) together with Remark 8, one obtains the corresponding H-theorem (3.27) for the approximate problem (4.34). Thus, we have shown that the approximate solutions $f_i^n \geq 0$ satisfy (4.2), uniformly in n.

Step 2

Velocity averaging Lemma 4.1 applied to $f_i^{n\delta} = (1/\delta)\log(1+\delta f_i^N)$, together with Lemmas 4.2–4.3, implies that, after passing to a subsequence, if necessary,

$$\int\limits_{\mathbb{R}^3} \phi f_i^n\,dv \xrightarrow[n\to\infty]{} \int\limits_{\mathbb{R}^3} \phi f_i\,dv \quad \text{strongly in } L^1((0,T)\times\Omega), \quad i = 1,2,3,4, \tag{4.53}$$

where f_i is a weak limit of $\{f_i^n\}_{n=1}^\infty$ in $L^1((0,T)\times\Omega\times\mathbb{R}^3)$.

Step3

Following very similar steps as for the original single specie Boltzmann equation (see, [8], or [14]), one shows, with the help of (4.53), the following *averaged continuity* of collision integrals that hold for all $\phi \in L^\infty((0,T)\times$

$\Omega \times \mathbb{R}^3)$ and $i = 1, 2, 3, 4,$

$$\frac{1}{\left(1 + \sum_{m=1}^{4} \int_{\mathbb{R}^3} f_m^n \, dv\right)} \int_{\mathbb{R}^3} \phi J_{in}^{E\pm}(\{f_i^n\}) \, dv \xrightarrow[n\to\infty]{} \frac{1}{\left(1 + \sum_{m=1}^{4} \int_{\mathbb{R}^3} f_m \, dv\right)} \int_{\mathbb{R}^3} \phi J_i^{E\pm}(\{f_i\}) \, dv \quad \text{in } L^1((0,T)\times\Omega) \tag{4.54}$$

and

$$\frac{1}{\left(1 + \sum_{m=1}^{4} \int_{\mathbb{R}^3} f_m^n \, dv\right)} \int_{\mathbb{R}^3} \phi J_{in}^{R\pm}(\{f_i^n\}) \, dv \xrightarrow[n\to\infty]{} \frac{1}{\left(1 + \sum_{m=1}^{4} \int_{\mathbb{R}^3} f_m \, dv\right)} \int_{\mathbb{R}^3} \phi J_i^{R\pm}(\{f_i\}) \, dv \quad \text{in } L^1((0,T)\times\Omega). \tag{4.55}$$

Remark 9. The *averaged continuity* (4.54)–(4.55) is also true for the original collision integrals $J_i^{E\pm}$ and $J_i^{R\pm}$, and for a sequence of renormalized solutions to (3.1), $\{f_i^n\}$, satisfying (4.2), uniformly in n.

The convergence in (4.54)–(4.55) together with the nonnegativity of f_i^n also implies for $\phi \in L^\infty((0,T)\times\Omega\times\mathbb{R}^3)$ and any $r > 0$,

$$L_{in}(\{f_i^n\}) \xrightarrow[n\to\infty]{} L_i(\{f_i\}) \qquad \text{in } L^1(0,T)\times\Omega), \qquad i = 1,2,3,4, \tag{4.56}$$

$$\int_{\mathbb{R}^3} \phi J_{in}^{E+}(\{f_i^n\}) \, dv \xrightarrow[n\to\infty]{} \int_{\mathbb{R}^3} \phi J_i^{E+}(\{f_i\}) \, dv \qquad \text{in measure on } (0,T)\times\Omega_r, \qquad i = 1,2,3,4, \tag{4.57}$$

$$\int_{\mathbb{R}^3} \phi J_{in}^{R+}(\{f_i^n\}) \, dv \xrightarrow[n\to\infty]{} \int_{\mathbb{R}^3} \phi J_i^{R+}(\{f_i\}) \, dv \qquad \text{in measure on } (0,T)\times\Omega_r, \qquad i = 1,2,3,4, \tag{4.58}$$

where

$$\Omega_r = \begin{cases} B_r, & \text{if } \Omega = \mathbb{R}^3; \\ \Omega, & \text{if } \Omega = [0,L]^3. \end{cases}$$

The passage to the limit is obtained in two steps. First, using similar techniques as in the proofs of (4.54)–(4.55) (see, [8] or [14]) together with the monotonicity property of $J_{in}^{E\pm}$ and $J_{in}^{R\pm}$ one shows that the function $\{f_i\}$ satisfies the inequality (super-solution property of $\{f_i\}$)

$$\begin{aligned} &f_i^{\#}(t,x,v) - f_i^{\#}(s,x,v) \exp\left\{-\left[\mathcal{F}_i^{\#}(t) - \mathcal{F}_i^{\#}(s)\right]\right\} \\ &\qquad \geq \int_s^t \left[J_i^{E+}(\{f_i\}) + J_i^{R+}(\{f_i\})\right]^{\#}(\tau,x,v) \exp\left\{-\left[\mathcal{F}_i^{\#}(t) - \mathcal{F}_i^{\#}(\tau)\right]\right\} d\tau, \end{aligned} \tag{4.59}$$

for any $0 < s < t \le T$ and a.e. in $(x,v) \in \Omega \times \mathbb{R}^3$, $i = 1,2,3,4$. Finally, by observing that for $i = 1,2,3,4$, $\delta > 0$, $n \ge 1$

$$(f_i^{n\delta})^\#(t,x,v) - (f_i^{n\delta})^\#(s,x,v) = \int_s^t \left(\left[\frac{J_{in}^{E+}(\{f_i^n\})}{1+\delta f_i^n} \right]^\# - \left[\frac{f_i^n}{1+\delta f_i^n} \right]^\# L_{in}(\{f_i^n\})^\# \right) d\tau, \tag{4.60}$$

and using (4.56), the weak convergence of $f_i^{n\delta}$ to f_i^δ, the weak convergence of $[J_{in}^{E+}(\{f_i^n\})]/(1+\delta f_i^n)$ and $h_i^{n\delta} = f_i^n/(1+\delta f_i^n)$ to some J_i^δ and h_i^δ, respectively, we obtain, after taking the weak limit in (4.60) as $n \to \infty$,

$$(f_i^\delta)^\#(t,x,v) - (f_i^\delta)^\#(s,x,v) = \int_s^t [(J_i^\delta)^\# - (h_i^\delta)^\# L_i(\{f_i\})^\# \, d\tau, \quad \text{a.e. in } (x,v) \in \Omega \times \mathbb{R}^3,\ i = 1,2,3,4. \tag{4.61}$$

From (4.29), $(f_i^\delta)^\# \xrightarrow[\delta \to 0^+]{} (f_i)^\#$ in $L^1(\Omega \times \mathbb{R}^3)$, uniformly in $t \in [0,T]$. Furthermore, since for $R > 0$,

$$0 \le z - \frac{z}{1+\delta z} \le \delta z R + z\chi_{z \ge R} \tag{4.62}$$

and $\{f_i^n\}$ is weakly relatively compact, one has

$$\sup_{0 \le t \le T} \|f_i - h_i^\delta\|_{L^1(\Omega\times\mathbb{R}^3)} \le \sup_{0 \le t \le T} \liminf_{n\to\infty} \|f_i^n - h_i^{n\delta}\|_{L^1(\Omega\times\mathbb{R}^3)} \xrightarrow[\delta \to 0^+]{} 0. \tag{4.63}$$

Finally, since $h_i^\delta \uparrow f_i$ as $\delta \downarrow 0^+$, the monotone convergence theorem implies (sub-solution property of $\{f_i\}$)

$$f_i^\#(t,x,v) - f_i^\#(s,x,v) \le \int_s^t [J_i^E(\{f_i\})^\#(\tau,x,v) + J_i^R(\{f_i\})^\#(\tau,x,v)] \, d\tau, \tag{4.64}$$

for $0 \le s \le t \le T$, $i = 1,2,3,4$, and a.e in $(x,v) \in \Omega \times \mathbb{R}^3$, if one can show that

$$J_i^\delta \le J_i^{E+}(\{f_i\}) + J_i^{R+}(\{f_i\}) \quad \text{a.e. in } (t,x,v) \in (0,T) \times \Omega \times \mathbb{R}^3. \tag{4.65}$$

Proof of (4.65) follows from the nonnegativity of J_{in}^{E+} and J_{in}^{R+}, and the application of the *averaged continuity* property (4.54)–(4.55). (see [8] or [14])

Remark 10. Super-solution property (4.59) of $\{f_i\}$ together with monotonicity in t of $\mathcal{F}_i(t)$ implies that for each $T > 0$ and a.e. in $(x,v) \in \Omega \times \mathbb{R}^3$, $J_i^{E+}(\{f_i\})$, $J_i^{R+}(\{f_i\}) \in L^1(0,T)$. The last fact combined with the sub-solution property (4.64) of $\{f_i\}$ shows that $J_i^{E-}(\{f_i\})$, $J_i^{R-}(\{f_i\}) \in L^1(0,T)$.

Step 4

The functions $\mathcal{F}_i^\#(t)$ defined in (4.47) is absolutely continuous in t for almost all $(x,v) \in \Omega\times\mathbb{R}^3$ and $d\mathcal{F}_i^\#/dt = L_i\{f_i\})^\#$, a.e. in t. The sub-solution property (4.64) of $\{f_i\}$ yields absolute continuity of $f_i^\#$ in t, for almost all $(x,v) \in \Omega \times \mathbb{R}^3$. Thus, $f_i^{\#} \exp \mathcal{F}_i^{\#}$ is also absolutely continuous in t, for almost all $(x,v)\Omega \times \mathbb{R}^3$ and from super-solution property (4.59) we obtain for $i = 1,2,3,4$,

$$\frac{d}{dt}\left(f_i^{\#} \exp \mathcal{F}_i^{\#}\right) \geq \left[J_i^{E+}(\{f_i\})^{\#} + J_i^{R+}(\{f_i\})^{\#}\right] \exp \mathcal{F}_i^{\#} \quad \text{a.e. in } t, \text{ for almost all } (x,v) \in \Omega \times \mathbb{R}^3, \tag{4.66}$$

or, for $i = 1,2,3,4$,

$$\frac{d}{dt} f_i^{\#} \geq J_i^{E}(\{f_i\})^{\#} + J_i^{R}(\{f_i\})^{\#} \quad \text{a.e. in } t, \text{ for almost all } (x,v) \in \Omega \times \mathbb{R}^3. \tag{4.67}$$

For $i = 1,2,3,4$, the inequality (4.67) is equivalent to

$$f_i^{\#}(t) - f_i^{\#}(s) \geq \int_s^t [J_i^{E}(\{f_i\})^{\#} + J_i^{R}(\{f_i\})^{\#}]\, d\tau \quad \text{for } 0 \leq s \leq t \text{ and for almost all } (x,v)\Omega \times \mathbb{R}^3. \tag{4.68}$$

Combination of (4.68) and (4.64) shows that $\{f_i\}$ is a mild solution of (3.1).

For the continuity property of $\{f_i\}$ we notice that (4.60) yields for $i = 1,2,3,4,\ 0 \leq s \leq t \leq T$ and $\delta > 0$

$$\|(f_i^{n\delta})^{\#}(t) - (f_i^{n\delta})^{\#}(s)\|_{L^1(\Omega\times\mathbb{R}^3)} \leq \int_s^t \left\| \frac{J_{in}^{E+}(\{f_i^n\})}{1+\delta f_i^n} \right\|_{L^1(\Omega\times\mathbb{R}^3)} d\tau. \tag{4.69}$$

Now, application of (4.69) together with (4.28) shows that for each $\nu > 0$ there exists $\tau > 0$ such that for $|t-s| \leq \tau$, and uniformly in n, one has

$$\|f_i^{n\#}(t) - f_i^{n\#}(s)\|_{L^1(\Omega\times\mathbb{R}^3)} \leq \nu. \tag{4.70}$$

After passing to the limit in (4.70) and observing that a norm is lower semicontinuous, one has $f_i^{\#} \in C([0,T]; L^1(\Omega \times \mathbb{R}^3)),\ i = 1,2,3,4$. Since the strongly continuous semigroup $U(t)$ is jointly continuous, one also has that $f_i \in C([0,T]; L^1(\Omega \times \mathbb{R}^3)),\ i = 1,2,3,4$. Note that for $f = (f_1, f_2, f_3, f_4),\ f^{\#}(t,x,v) = (U(-t)f)(t,x,v)$. □

Remark 11. The mild solution obtained in Theorem 4.2 obeys the conservation of mass and momentum (3.20)–(3.21). Instead of the conservation of energy property (3.22), one obtains

$$\sum_{i=1}^{4} \iint\limits_{\Omega\times\mathbb{R}^3} \left(\frac{m_i v^2}{2} + E_i\right) f_i(t,x,v)\, dv dx \leq \sum_{i=1}^{4} \iint\limits_{\Omega\times\mathbb{R}^3} \left(\frac{m_i v^2}{2} + E_i\right) f_{i0}(x,v)\, dv dx. \tag{4.71}$$

This is due to lack of higher moments estimations for J_i^E and J_i^R, and the fact the basic estimation (4.2) (see also Remark 7) guarantees weak compactness of the sequence $\{(1+|x|^k+|v|^k) f_i^n\}$ in $L^1((0,T)\times\Omega\times\mathbb{R}^3)$ for $0 \leq k < 2$, not including $k = 2$.

I note that when the steric factors $\beta_{ij} = 0$ for $i, j = 1, 2, 3, 4,$ system (3.1) reduces itself to the hard-sphere Boltzmann equation for non-reacting mixtures. Thus, we also have

Corollary 4.1. *Assume that the assumptions of Theorem 4.2 are satisfied and, in addition,* $\beta_{ij} = 0$ *for* $i, j = 1, 2, 3, 4.$ *Then there exists a nonnegative mild solution* f_i *to the hard-sphere Boltzmann equation for non-reacting mixtures, with* $f_i \in C([0,T]; L^1(\Omega \times \mathbb{R}^3))$ *satisfying* (4.3)*, and such that* $f_i(t)\big|_{t=0} = f_{i0}$*, for* $i = 1, 2, 3, 4.$

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