On Some Open Problems in the Revised Enskog Equation for Dense Gases

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1 The closure relation for the revised Enskog equation

Consider a gas composed of identical hard spheres of diameter a, i.e., with the potential interaction given by

$$\phi^{HS}(r) = \begin{cases} \infty, & \text{if } r < a; \\ 0, & \text{if } r \geq a. \end{cases} \tag{1.1}$$

When two particles of equal mass collide, their velocities v_1 and v_2 take post-collisional values

$$v_1' = v_1 - \epsilon\left(\epsilon \cdot (v_1 - v_2)\right), \tag{1.2}$$

$$v_2' = v_2 + \epsilon\left(\epsilon \cdot (v_1 - v_2)\right). \tag{1.3}$$

Here, ϵ is a vector along the line passing through the centers of the spheres at their impact, i.e. $\epsilon \in \mathbb{S}_+^2 = \{\epsilon \in \mathbb{R}^3 : |\epsilon| = 1\,,\ \epsilon \cdot (v_1 - v_2) \geq 0\}$, where $(\cdot,\cdot)$ is the scalar product in $\mathbb{R}^3$.

Within the kinetic theory the state of fluid depends upon (among other things) the one–particle distribution function $f_1(t, r_1, v_1)$. f is the probability density, at time t, of finding a particle at the point r_1 with velocity v_1). The exact rate of change of f_1 is given by (see, for example, [1])

$$\frac{\partial f_1}{\partial t} + v_1 \cdot \frac{\partial f_1}{\partial r_1} = \int dr_2 dv_2\, K_{12} f_2(t, r_1, v_1, r_2, v_2), \tag{1.4}$$

with

$$K_{12} f_2 = (N-1)a^2 \int\limits_{\mathbb{S}_+^2} \Big[f_2(t, r_1, v_1', r_2, v_2')\delta(r_{12} - a\epsilon) \tag{1.5}$$

$$- f_2(t, r_1, v_1, r_2, v_2)\delta(r_{12} + a\epsilon) \Big] \epsilon \cdot (v_1 - v_2)\, d\epsilon, \tag{1.6}$$

and $r_{12} = r_1 - r_2$. The probability density of pairs of particles in collisional configurations is described by the two–particle distribution function f_2. The above equation is the first of the infinite BBGKY-hierarchy for hard spheres.

In the past several different approaches have been used to close equation (1.4). Usually, one approximates f_2 using the relation

$$f_2(t, r_1, v_1, r_2, v_2) = Y_2(t, r_1, v_1, r_2, v_2)\, f_1(t, r_1, v_1)\, f_1(t, r_2, v_2), \tag{1.7}$$

with Y_2 chosen in a suitable way. In the case of the revised Enskog equation ([2])

$$Y_2 = g_2(r_1, r_2 \,|\, n(t)), \tag{1.8}$$

where g_2 is the pair correlation function for the system of hard spheres in ***non–uniform equilibrium***, $n(t, r_1) = \int f_1(t, r_1, v_1)\, dv_1$ is the local density, and

$$a \to 0, \quad N \to \infty, \quad \text{(dense gas limit)} \quad Na^3 \to \mathit{const} > 0. \tag{1.9}$$

The notation $g_2(r_1, r_2 \,|\, n(\cdot))$ indicates that g_2 is a functional of n (see, (1.12) below).

For comparison, the closure relation for the Boltzmann equation has the form

$$Y_2 = 1 \tag{1.10}$$

with

$$a \to 0, \quad N \to \infty, \quad \text{(dilute gas limit)} \quad Na^2 \to \mathit{const} > 0. \tag{1.11}$$

Remark 1.1. *Let us note that for the Boltzmann equation* $Na^3 \to 0$.

With the change $f_1 \to N f_1$, $f_2 \to N(N-1) f_2$, etc., g_2, defined by $f_2 = g_2\, f_1\, f_1$, has the form ([3])

$$\begin{aligned} g_2(r_1, r_2 \,|\, n(t)) = \exp(-\beta\phi^{HS}(|r_1 - r_2|)) \Big\{ 1 + \int V(12 \,|\, 3) n(t, r_3)\, dr_3 \\ + \frac{1}{2} \int\int V(12 \,|\, 34) n(t, r_3)\, n(t, r_4)\, dr_3 dr_4 + \cdots \\ + \frac{1}{(k-2)!} \int dr_3 \cdots \int dr_k\, n(3) \cdots n(k)\, V(12 \,|\, 3 \ldots k + \cdots \Big\}, \end{aligned} \tag{1.12}$$

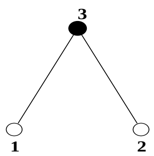
where $n(k) = n(t, r_k)$, $\beta = 1/k_B T$, $V(12 \,|\, 3 \ldots k)$ is the sum of all graphs of k labeled points which are biconnected when the Mayer factor $f_{ij} = \exp(-\beta\phi^{HS}(|r_i - r_j|)) - 1$ is added. Observe that in the particular case of the hard-spheres potential, the Mayer factors, f_{ij}, become

$$f_{ij} = \Theta(|r_i - r_j| - a) - 1, \quad \text{with} \quad \Theta \quad \text{the step function.} \tag{1.13}$$

For example, the second term in the curly brackets on the right side of (1.12) has the form

$$\int n(r_3)\, V(12 \,|\, 3)\, dr_3 = \int n(r_3)\, f_{13}\, f_{23}\, dr_3. \tag{1.14}$$

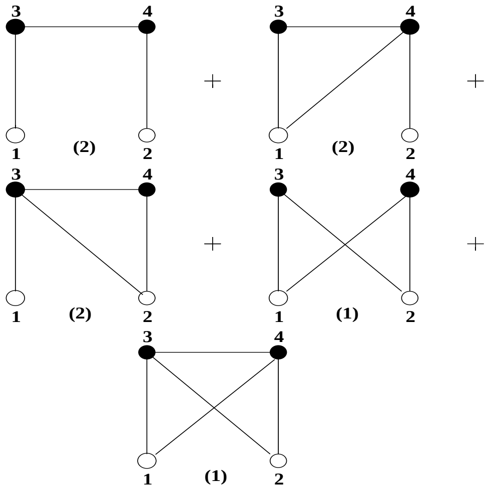
The right hand side of (1.14) can also be represented in the following graphical way:



The next coefficient is

$$\int n(r_3)n(r_4)\, V(12\,|\,34)\, dr_3 dr_4 =$$
$$\int n(r_3)n(r_4)\Big[(2)f_{13}f_{34}f_{24} + (2)f_{13}f_{34}f_{14}f_{24}\Big]dr_3dr_4 +$$
$$\int n(r_3)n(r_4)\Big[(2)f_{13}f_{34}f_{24}f_{23} + (1)f_{13}f_{24}f_{14}f_{23}\Big]dr_3dr_4$$
$$+\int n(r_3)n(r_4)\Big[(1)f_{13}f_{34}f_{24}f_{14}f_{23}\Big]dr_3dr_4,$$

with its graphical representation



and similarly for higher order coefficients.

Thus, the revised Enskog equation can be written in the form

$$\frac{\partial f}{\partial t} + v_1 \frac{\partial f}{\partial r_1} = E(f) = E^+(f) - E^-(f), \tag{1.15}$$

with

$$E^+(f) = a^2 \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2_+} g_2(r_1, r_1 - a\epsilon\,|\,n(t)) f(t, r_1, v_1')\, f(t, r_1 - a\epsilon, v_2')\, \epsilon\cdot(v_1 - v_2)\, d\epsilon dv_2 \tag{1.16}$$

and

$$E^-(f) = a^2 \iint\limits_{\mathbb{R}^3\times\mathbb{S}^2_+} g_2(r_1, r_1 + a\epsilon\,|\,n(t)) f(t, r_1, v_1)\, f(t, r_1 + a\epsilon, v_2)\, \epsilon\cdot(v_1 - v_2)\, d\epsilon dv_2 \tag{1.17}$$

2 The H–function for the revised Enskog

The H–function for the revised Enskog equation is different from the corresponding H-function for the Boltzmann equation. It has the form ([4])

$$H_{RE}(t) = \iint f(t, r_1, v_1) \log f(t, r_1, v_1)\, dv_1 dr_1 + H_{corr}(t), \tag{2.1}$$

The correlational part $H_{corr}(t)$ is expressed in terms of the Mayer graphs ([5])

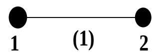
$$H_{corr}(t) = \sum_{k=2}^{\infty} \frac{1}{k!} \int dr_1 \cdots \int dr_k\, n(1) \dots n(k)\, V(1 \dots k), \tag{2.2}$$

where $V(1 \dots k)$ is the sum of all irreducible Mayer graphs which doubly connect k particles. In contrast to Botzmann's H–function, H_{RE} consists of two parts: the kinetic part ($H_B(t)$) and the correlational part. $-H_{RE}(t)$ (modulo an additive constant) is the equilibrium (non-uniform) entropy for the infinity system of hard–spheres. In addition, one recognizes that

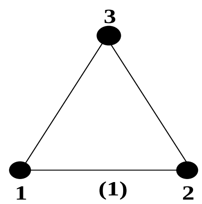
$$-\beta H_{corr} = \mathcal{A}^{excess} \tag{2.3}$$

is the excess free energy for the system hard-spheres in the **non-uniform** equilibrium. The first three terms of H_{corr} together with their graphical representations are given below.

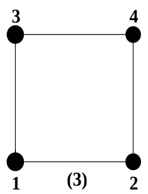
$$\frac{1}{2!} \int n(r_1)\, n(r_2)\, V(12)\, dr_1 dr_2 = \frac{1}{2!} \int n(r_1)\, n(r_2)\, f_{12}\, dr_1 dr_2$$



$$\frac{1}{3!} \int n(r_1)\, n(r_2)\, n(r_3)\, V(123)\, dr_1 dr_2 dr_3 = \frac{1}{3!} \int n(r_1)\, n(r_2)\, n(r_3)\, f_{12} f_{23} f_{13}\, dr_1 dr_2 dr_3$$



$$\frac{1}{4!} \int n(r_1)\, n(r_2)\, n(r_3)\, n(r_4)\, V(1234)\, dr_1 dr_2 dr_3 dr_4 =$$



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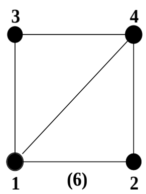

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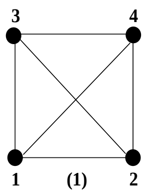

$$= \frac{1}{4!} \int n(r_1)n(r_2)n(r_3)n(r_4)dr_1dr_2dr_3dr_4 \times$$
$$\Big\{ (3) f_{12}f_{23}f_{34}f_{14} + (6) f_{12}f_{23}f_{34}f_{14}f_{13} + (1) f_{12}f_{23}f_{34}f_{14}f_{13}f_{24} \Big\}$$

Next, I list important differences between solutions for the Boltzmann and revised Enskog equations. If $H_B(t)$ denotes Boltzmann's H-function

$$H_B(t) = \iint f(t, r_1, v_1) \log f(t, r_1, v_1)\, dv_1 dr_1 \tag{2.4}$$

and $E_B(f) = E_B^+(f) - E_B^-(f)$ is the Boltzmann collision operator corresponding to the hard-sphere potential with $E_B^{\pm}$ given by

$$E_B^+(f) = a^2 \iint_{\mathbb{R}^3 \times \mathbb{S}^2_+} f(t, r_1, v_1')\, f(t, r_1, v_2')\, \epsilon \cdot (v_1 - v_2)\, d\epsilon dv_2 \tag{2.5}$$

and

$$E_B^-(f) = a^2 \iint_{\mathbb{R}^3 \times \mathbb{S}^2_+} f(t, r_1, v_1)\, f(t, r_1, v_2)\, \epsilon \cdot (v_1 - v_2)\, d\epsilon dv_2, \tag{2.6}$$

and f is a solution of the Boltzmann equation then:

$$\frac{d}{dt} H_B(f) = 0 \quad \Longleftrightarrow \quad f = \omega(n(t, r_1), u(t, r_1), T(t, r_1)) \quad \Longleftrightarrow \quad E_B(f) = 0, \tag{2.7}$$

where

$$\omega(n(t, r_1), u(t, r_1), T(t, r_1)) = \frac{n(t, r_1)}{(2\pi k_B T E(t, r_1))^{3/2}} \exp\left(-\frac{|v - u(t, r_1)|^2}{2k_B T(t, r_1)} \right) \tag{2.8}$$

is a local Maxwellian uniquely determined by macroscopic density $n(t, r_1)$, macroscopic velocity $u(t, r_1)$, and macroscopic temperature $T(t, r_1)$.

Relations (2.7) look different in the case of the revised Enskog equation:

$$\frac{d}{dt} H_E(f)(t) = 0 \quad \Longleftrightarrow \quad f = \omega(n(t, r_1), \overbrace{u(t), T(t)}^{!!!}) \tag{2.9}$$

and

$$E(f) = 0 \quad \Longleftrightarrow \quad f = \omega(\overbrace{n(t), u(t), T(t)}^{!!!}) \tag{2.10}$$

The exclamation points indicate the differences with the Boltzmann equation.

In contrast to the Boltzmann equation, the fluid underlying the revised Enskog equation is **NOT** an ideal fluid. Its equilibrium equation of state expressed in terms of the virial coefficients B_k has the form

$$\frac{p}{nk_B T} = 1 + \sum_{k=1}^{\infty} n^k B_{k+1}. \tag{2.11}$$

For example, the second virial coefficient ($k = 1$) is equal to $2/3\pi a^3$.

Even if in the cluster Mayer expansion of g_2 (1.12) we keep only the first term, i.e., $g_2 = \exp(-\beta\phi^{HS}(|r_1 - r_2|))$ (and obtain so called Boltzmann-Enskog equation), the corresponding equation of the state,

$$\frac{p}{nk_BT} = 1 + \frac{2}{3}\pi a^3 n, \tag{2.12}$$

differs from the equation of state for the ideal gas.

3 Open problems

3.1 Convergence of the cluster Mayer expansion for g_2

The first natural problem in the RET concerns the convergence of the series in (1.12). In the homogeneous case, i.e., when n does not depend on the spatial variable r, the series in (1.12) is convergent for sufficiently small densities n ([6]). It is not clear whether this result can be extended to the non-homogeneous case. But even if that is the case, there are still difficulties in applying this fact to the existence theorems for the RET. It is not clear whether the RET preserves the boundedness of the density with time evolution. Recently, together with G. Stell, I elaborated further on this topic in [7] and its relation to Resibois' work [4].

The existence theorems obtained so far dealt only with the modifications of the revised Enskog equation. In [8] and [9] the authors obtained small initial data and vacuum solutions with Y_2 in (1.7) assumed to be smooth and bounded for smooth and bounded densities. This condition cannot be checked for the series in (1.12) before its convergence is confirmed. The authors in [10], [11], [12], and [13] have obtained renormalized solutions in L^1, using the weak compactness method, originally applied to the Boltzmann equation by DiPerna and Lions [14]. They consider either the cases of $Y_2 \equiv 1$ (the Boltzmann–Enskog equation) or the cases of Y_2 that are bounded and with compact support. In [7] and [15] some progress has been made by considering Y_2 of the form:

$$\begin{aligned} Y_2 = {} & \exp\left(-\beta\phi^{HS}(|r_1 - r_2|)\right) \times \\ & \left\{1 + \sum_{k=3}^{i} \frac{1}{(k-2)!} \int dr_3 \cdots \int dr_k\, n(3) \ldots n(k)\, V(12\,|\,3\ldots k)\right\}, \end{aligned} \tag{3.1}$$

for $i \geq 2$, and with the convention that $Y_2 = \exp\left(-\beta\phi^{HS}(|r_1 - r_2|)\right)$ for $i = 2$. The case of $i = 2$ corresponds to the Boltzmann–Enskog equation, i.e., the case when $Y_2 \equiv 1$. It has been shown in [7] that the contact value of Y_2 from (3.1) is bounded in L^∞ as long as n is bounded in L^1. This a priori bound implies the boundedness of the configurational part of the Liapunov functional that makes the use of the weak compactness method possible. I anticipate that the above mentioned a priori estimation formally holds when $i = \infty$, thus promising a positive answer to the existence theorem in L^1 for the RET.

Below is an example of the existence theorem for the truncated version of the revised Enskog equation with Y_2 from (1.7) given in (3.1).

Theorem 3.1. *If an initial value $f_0 \geq 0$ satisfies*

$$\iint\limits_{\Omega\times\mathbb{R}^3} (1 + v_1^2 + r_1^2 + |\log f_0(r_1, v_1)|) f_0(r_1, v_1)\, dv_1 dr_1 \leq C_0, \tag{3.2}$$

then there exists a nonnegative renormalized solution f of the ***truncated*** *revised Enskog equation on $[0, \mathcal{T}]$ and any $\mathcal{T} > 0$.*
Furthermore, for some sequence $\{t_n\}$ with $t_n \underset{n\to\infty}{\to} \infty$

$$f(t_n, r_1, v_1) \underset{n\to\infty}{\to} \omega(r_1, v_1), \text{ almost everywhere in } r_1,\ v_1, \tag{3.3}$$

where $\omega(r_1, v_1) = \frac{n(r_1)}{(2\pi k_B X)^{3/2}} \exp\left(-\frac{|v_1-u|^2}{2k_B X}\right)$.
Here, $\Omega = \mathbb{R}^3$ or $\Omega = [0, L]^3$ (plus periodic conditions). In the latter case r_1^2 appearing in (3.2) is superfluous.

Remark 3.1. *Here $n(r_1)$ and u are macroscopic density and velocity of the gas, respectively. However, X may not not necessary the macroscopic temperature of the gas. This due to the fact that the convergence above is in $L^1[(1 + |v_1|)^k]$ for only $0 \leq k < 2$, not including $k = 2$.*

3.2 Equilibrium solutions

Let us consider the local Maxwellian $\omega(r_1, v_1)$, obtained in (3.3) in the case $\Omega = [0, L]^3$. Then ω satisfies the stationary revised Enskog equation

$$v_1 \frac{\partial \omega}{\partial r_1} = E(\omega), \quad \text{with} \quad \omega(r_1) \quad \text{periodic on} \quad [0, L]^3 \tag{3.4}$$

where

$$\omega = \frac{n(r_1)}{(2\pi k_B X)^{3/2}} \exp\left(-\frac{|v_1 - u|^2}{2k_B X}\right). \tag{3.5}$$

Before, I proceed further it is important to notice that in the case of the Boltzmann equation

$$v_1 \frac{\partial \omega}{\partial r_1} = E_B(\omega) \equiv 0, \quad \text{(property of (2.7))} \tag{3.6}$$

and since $\omega(r_1)$ is periodic on $[0, L]^3$ one obtains that $n(r_1) \equiv$ *constant*, if one chooses $u = 0$ for the equilibrium macroscopic velocity.

However, for the revised Enskog equation, again with $u = 0$, (3.4) reduces to

$$\frac{\partial \log n}{\partial r_1} = \int\limits_{\Omega} \left(\frac{\partial f_{12}}{\partial r_1}\right) \overline{g}_2(r_1, r_2 \,|\, n)\, n(r_2)\, dr_2, \quad \text{where,} \tag{3.7}$$

$$\overline{g}_2 = 1 + \sum_{k=3}^{\infty} \frac{1}{(k-2)!} \int\limits_{\Omega} dr_3 \cdots \int\limits_{\Omega} dr_k\, n(3) \ldots n(k)\, V(12\,|\,3 \ldots k), \tag{3.8}$$

with $n(k) = n(r_k)$ and f_{ij} given in (1.13).

Equation (3.7) (in fact, a system of three equations) is the first member of the BBGY equilibrium hierarchy for the system of hard-spheres ([16]). Since $\overline{g}_2$ is expressed only in terms of n, the above equation is **closed** and **exact**.

From the mathematical point of view nothing is known about equation (3.7). I simplify the problem by considering only the first two terms of $\overline{g}_2$ expansion:

$$\overline{g}_2 \approx 1 + \int_\Omega f_{13} f_{23} n(t, r_3)\, dr_3. \tag{3.9}$$

Then, equation (3.7) has the form

$$\frac{\partial \log n}{\partial r_1} + \frac{\partial}{\partial r_1}\Bigg(\int_{B(r_1,a)} G(r_1, r_2 \,|\, n)\, n(r_2)\, dr_2 \Bigg) = 0, \tag{3.10}$$

with $B(r_1, a) = \{r_2 \in \Omega : |r_1 - r_2| \leq a\}$ and

$$G = 1 + \frac{1}{2} \int_\Omega f_{13} f_{23} n(t, r_3)\, dr_3. \tag{3.11}$$

Observe that $G \in L^\infty(\Omega \times \Omega)$. Thus, after integrating the above equation, we obtain that $n \in L^1(\Omega) \cap L^\infty(\Omega)$.

Finally, equation (3.10) is equivalent to

$$n(r_1) = N \frac{\exp\Bigg(- \int_{B(r_1,a)} G(r_1, r_2 \,|\, n)\, n(r_2)\, dr_2 \Bigg)}{\int_\Omega \exp\Bigg(- \int_{B(r_1,a)} G(r_1, r_2 \,|\, n)\, n(r_2)\, dr_2 \Bigg) dr_1}, \tag{3.12}$$

where $N = \int_\Omega n(r_1)\, dr_1$ and $\Omega = [0, L]^3$. The following result holds:

Theorem 3.2.

(1) If $\dfrac{N}{\textit{vol}(\Omega)} a^3$ *is small enough then equation* (3.12) *has a unique solution* $n(r_1) =$ ***const****.*

(2) For some values of the constant density, $\dfrac{N}{\textit{vol}(\Omega)}$*, and the diameter of hard spheres, a, equation* (3.12) *has a* ***non-constant*** *periodic solution in* $L^1(\Omega) \cap L^\infty(\Omega)$*.*

From Theorem 3.1 we obtain, for some sequence $\{t_n\}$ with $t_n \underset{n\to\infty}{\longrightarrow} \infty$,

$$f(t_n, r_1, v_1) \underset{n\to\infty}{\longrightarrow} \frac{n(r_1)}{(2\pi k_B X)^{3/2}} \exp\left(-\frac{(v_1)^2}{2k_B X} \right) \quad \text{pointwise almost everywhere}, \tag{3.13}$$

where $n(r_1)$ a solution of (3.12). It is easy to check that a constant density is always a solution of (3.12). However, Theorem 3.2 implies that for suitable choices of N, vol(Ω), and a there exists non–constant periodic solution $n(r_1)$. For such choices of N, vol(Ω), and a the following question arises:

Is $\mathbf{n(r_1)}$ in (3.13) a constant or periodic function of $\mathbf{r_1}$?

If the revised Enskog equation drives the initial distribution to the local Maxwellian with periodic n, then this property could be view as a possible phase transition of the system of hard spheres. At the same time one should be caution that the result (2) of Theorem 3.2 has been only proven for the truncated $\overline{g}_2$ given in (3.9) and not for the full $\overline{g}_2$ in (3.8).

4 The Euler level for the revised Enskog equation

Recently, in [17], [18], and [19] some progress has been reported for the hydrodynamic limit from the Boltzmann equation. M. Lachowicz in a series of works (see [20] and references there) has studied some variants of the standard Enskog equations (albeit with not physical Y_0) or the Boltzmann-Enskog equations. In contrast to the work in [18] for the Boltzmann equation, M. Lachowicz presents convergence for very short times (comparable to the mean time between collisions). I believe that in order to make substantial progress in this direction for the full revised Enskog equation (with g_2 given in (1.12)) the convergence of the Mayer cluster expansion in (1.12) and additional (spectral) properties of the linearized Enskog operator.

Below, I consider an analog of the result in [17] (proven there for the Boltzmann equation) restricted here to the Euler level. The usual time-space scaling (hyperbolic or Euler scaling)

$$t \longmapsto \frac{t}{\delta} \qquad x \longmapsto \frac{x}{\delta}, \qquad \delta = \text{mean free path}, \tag{4.1}$$

does not work for the revised Enskog equation due to existence of the factor $g_2(r, r \pm a\epsilon) \neq 1$. In addition we have to deal with two small parameters:

$$a \to 0 \quad \text{(hard-spheres' diameter)} \quad \text{and} \quad \delta \to 0 \quad \text{with } \delta \approx 1/(na^2) \tag{4.2}$$

If we recall that the Enskog theory is valid in the limit

$$a \to 0 \quad n \to \infty, \quad \text{(dense gas limit)} \quad na^3 \to \mathit{const} > 0. \tag{4.3}$$

then the following scaling is of some interest:

$$f \longmapsto \frac{f}{\delta^3} \qquad \text{with} \qquad \delta \approx a^m, \quad m > 0. \tag{4.4}$$

Below, for simplicity, consider only the followng relation between a and δ: $\delta \approx a^m,\ m > 0$.

Consider the revised Enskog equation with the scaling $f = (1/\delta^3)F$

$$\frac{\partial F}{\partial t} + v \cdot \frac{\partial F}{\partial r} = E_\delta(F) = \frac{a^2}{\delta^3} \iint\limits_{\mathbb{R}^3 \times \mathbb{S}^2_+} \left[F' F'_- Y^\delta_- - F F_+ Y^\delta_+\right] \epsilon \cdot (v - w)\, d\epsilon dw, \tag{4.5}$$

where $F'_- = F(t, r - a\epsilon, w')$, $F_+ = F(t, r + a\epsilon, w)$, $F' = F(t, r, v')$, $n(t, r) = \int_{\mathbb{R}^3} F(t, r, v)\, dv$, and $Y^\delta_\pm = g_2\left(r, r \pm a\epsilon \,|\, (1/\delta^3)n\right)$, with g_2 given in (1.12).

Lemma 4.1. *For $n \in L^1$ and $\delta = (a/d)^m$, with $0 < m \le 1$ and $d > 0$,*

$$\lim_{a \to 0,\, \delta \to 0} Y_\pm^\delta(r, r \pm a\epsilon) = \begin{cases} 1, & 0 < m < 1 \\ Y_0(n(r)), & m = 1 \end{cases} \tag{4.6}$$

almost everywhere in r, where $Y_0(n(r))$ is the radial pair correlation function for the system of hard spheres with diameter d in (uniform) equilibrium with density $n(r)$ given by (1.12) *with $|r_1 - r_2| = d$, i.e., $Y_0(n(r)) = 1 + (2\pi d^3/3)(5/8)n(r) + \cdots$.*

The above lemma is obtained under the assumption that the Mayer cluster expansion (1.12) converges. I have indicated earlier the convergence of (1.12) has not been yet proven.

For a solution F of (4.5) and a given function $\varphi(t, r, v)$, define

$$n = \int_{\mathbb{R}^3} F\, dv \qquad \langle \varphi \rangle = (1/n) \int_{\mathbb{R}^3} \varphi F\, dv. \tag{4.7}$$

Proposition 4.1 (Transfer equation). *For any $\varphi(t, r, v)$,*

$$\frac{\partial}{\partial t} n\langle \varphi \rangle + \frac{\partial}{\partial r} \cdot n\langle v\varphi \rangle - n\langle v \cdot \frac{\partial \varphi}{\partial r} \rangle = \frac{\partial}{\partial r} J^P + \sigma_s, \tag{4.8}$$

with

$$\sigma_s = \frac{a^2}{2\delta^3} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2_+} F F_+ Y_+^\delta \left[\varphi' + \varphi'_+ - \varphi - \varphi_+\right] \epsilon \cdot (v - w)\, d\epsilon dv dw, \tag{4.9}$$

$$J^P = -\frac{a^3}{2\delta^3} \int_0^1 \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2_+} F(t, r - \alpha a\epsilon, v) F(t, r + (1-\alpha) a\epsilon, w) \cdot \tag{4.10}$$

$$g_2^{\alpha,\delta} \Big[\varphi(r - \alpha a\epsilon, v') - \varphi(r - \alpha a\epsilon, v)\Big] (\epsilon \cdot (v - w))\, \epsilon d\epsilon dv dw, \tag{4.11}$$

and $g_2^{\alpha,\delta} = g_2\left(r - \alpha a\epsilon, r + (1-\alpha)a\epsilon \,|\, (1/\delta^3)n\right)$.

Remark 4.1. *The flux term $\partial J_P/\partial r$ in the right hand side of* (4.8) *does **not** appear in the corresponding transfer equation for the Boltzmann equation.*

The next theorem is the revised Enskog version of Theorem 1 in [17] (presented there for the Boltzmann equation).

Theorem 4.1. *For $\delta = (a/d)^m$, $d > 0$, and $2/3 < m \le 1$, let $F_\delta(t, r, v)$ be a sequence of (renormalized) nonnegative solutions of the problem*

$$\frac{\partial F_\delta}{\partial t} + v \cdot \frac{\partial F_\delta}{\partial r} = E_\delta(F_\delta). \tag{4.12}$$

Assume that $F_\delta \to F_0$ a.e. and that the moments $\langle F_\delta \rangle$, $\langle vF_\delta \rangle$, $\langle v \otimes vF_\delta \rangle$, and $\langle v|v|^2 F_\delta \rangle$ converge in the sense of distribution to the corresponding moments of F_0.

In addition, let us assume that

$$\lim_{\delta\to 0}\langle \varphi F_\delta \log F_\delta\rangle = \langle \varphi F_0 \log F_0\rangle, \qquad \textit{for } \varphi = 1,\ v,\ \textit{and} \tag{4.13}$$

$$\limsup_{\delta\to 0}(\delta^3/a^2)\langle E_\delta(F_\delta)\log F_\delta\rangle \le \langle E_B(F_0)\log F_0\rangle, \tag{4.14}$$

where $E_B = E_B^+ - E_B^-$ *is the Boltzmann operator* (2.5)–(2.6) *with the diameter* d.
Then

$$F_0(t,r,v) = \frac{n(t,r)}{(2\pi k_B T(t,r))^{3/2}} \exp\left(-\frac{|v-u(t,r)|^2}{2k_B T(t,r)}\right). \tag{4.15}$$

Furthermore, when $2/3 < m < 1$ *the functions* n, u, *and* T *appearing in* (4.15) *satisfy the compressible Euler equations (perfect fluid).*
Finally, when $m = 1$, *the functions* n, u, *and* T *are solutions of the following system*

$$\frac{\partial n}{\partial t} + \nabla_r\cdot(nu) = 0 \tag{4.16}$$

$$\frac{\partial(nu)}{\partial t} + \nabla_r\cdot(nu\otimes u) + \nabla_r\cdot\left(nk_BT + \underbrace{\frac{2}{3}\pi d^3 Y_0(n) n^2 k_B T}\right) = 0 \tag{4.17}$$

$$\frac{\partial}{\partial t}\left[n\left(\frac{1}{2}|u|^2 + \frac{3}{2}k_BT\right)\right] + \nabla_r\cdot\left[nu\left(\frac{1}{2}|u|^2 + \frac{5}{2}k_BT\right)\right] + \underbrace{\nabla_r\cdot(\frac{2}{3}\pi d^3 Y_0(n) n^2 k_B T u)} = 0 \tag{4.18}$$

and the entropy inequality

$$\frac{\partial}{\partial t}\left[n\left(\log\frac{n}{T^{3/2}} + \underbrace{\frac{2}{3}\pi d^3\int_0^n Y_0(\rho)\,d\rho}\right)\right] + \nabla_r\cdot\left[nu\left(\log\frac{n}{T^{3/2}} + \underbrace{\frac{2}{3}\pi d^3\int_0^n Y_0(\rho)\,d\rho}\right)\right] \le 0. \tag{4.19}$$

In particular, the pressure $p = nk_BT + \underbrace{\frac{2}{3}\pi d^3 Y_0(n) n^2 k_B T}$.

Curly brackets indicate the additional terms not appearing in the Euler equations for the Boltzmann equation.

Corollary 4.1 (Relation between free energy and $\mathbf{Y_0(n)}$).

$$\frac{2}{3}\pi d^3\int_0^n Y_0(\rho)\,d\rho = \sum_{k=1}^{\infty} n^k \overline{B}_{k+1} = \frac{\mathcal{A}_{ex}}{k_BT},$$

where $\overline{B}_k = -\frac{1}{k-1}B_k$ *is expressed in term of virial coefficients, and* $\mathcal{A}_{ex}$ *is the equilibrium excess free energy per particle of of hard-spheres systems.*

Note that the equilibrium pressure for hard-spheres systems is given by

$$\frac{p}{nk_BT} = 1 + \sum_{k=1}^{\infty} n^k B_{k+1}. \tag{4.20}$$

By considering the excluded volume of hard spheres (recall the closure relation for f_2 in (1.7) and (1.8)), the revised Enskog equation, in contrast to the Boltzmann equation, does not ignore the difference in position between colliding particles. Therefore, collisional transfer of momentum and energy from the center of one colliding sphere to the other are taken into account. Thus the differences in the transfer equations between the revised Enskog and Boltzmann equations (see (4.8) and Remark 4.1). These transfers take place over distances proportional to a, the diameter of hard spheres. This is to be contrasted with convective transport over a distance of order δ (the mean free path between collisions).

As a consequence, convective transport dominates at low densities, where $\delta/a \gg 1$. On the other hand, at high densities $\delta/a \leq 1$, and collisional transfer dominates the transport. The mean free path $\delta \approx 1/(na^2)$ (at least for dilute gases), and therefore the case of $m = 1$, i.e., $\delta \approx a$, corresponds to $na^3 \approx 1$. This is in complete agreement with the initial scaling $f = (1/\delta^3)F$.

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