Kinetic Theory of Atoms and Photons: An Application to the Milne–Chandrasekhar Problem

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Abstract. We consider a kinetic model describing the interactions between monochromatic photons and gas–particles with two internal energy levels. Using the moment equations we formulate so–called Milne–Chandrasekhar problem. Next, we provide the existence and uniqueness theorems for the resulting system of partial differential equations.

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1. Introduction In a recent paper [RSM], Rossani, Spiga and Monaco have proposed the kinetic model for the study of two–level atoms and monochromatic photons. The model incorporates a basic feature that allows a good description of the problem, namely the interplay of inelastic collisions between atoms, on one side, and interaction between gas and radiation, on the other. The most remarkable result of such a modelling is that, under thermodynamical equilibrium conditions, Planck's law of radiation is recovered selfconsistently, without resorting to additional hypotheses.

An interesting application for the above kinetic model is the Milne–Chandrasekhar problem [CHA, MIL]. A slab is filled with a gas, illuminated from one side. The problem consists in the study of the evolution, within the slab, of radiation field, excited atom density and gas temperature.

In this report, after a brief description of the model, we derive the corresponding moment equations for the Milne–Chandrasekhar problem and state existence and uniqueness theorems for the resulting system of partial differential equations. In [MPR], we provide further details on approximate analytical solution to the stationary problem, as well as numerical solutions to the time dependent and stationary problems.

2. The Kinetic Model

Consider the following physical system:

(a) A gas of atoms A of mass m endowed with only two internal energy levels E_1 and E_2, $E_1 < E_2$. In what follows we will denote by A_1 and A_2, respectively, particles A at levels "1" and "2" (fundamental and excited levels).

(b) A radiation field of photons p at a fixed frequency

$$\nu = \frac{\Delta E}{h} \ , \qquad \Delta E = E_2 - E_1 \ ,$$

h being the Planck constant, interacting with the above gas–particles.

The following interactions between particles and between particles and photons take place:

a) Elastic interactions among particles with the same or different energy level.

b) Inelastic mechanism of exchange between the internal energy levels

$$A_1 + A_1 \rightleftharpoons A_2 + A_1 \ ,$$

$$A_1 + A_2 \rightleftharpoons A_2 + A_2 \ .$$

c) Interactions between gas–particles and photons

c.1) Absorption

$$A_1 + p \rightarrow A_2 \ ,$$

c.2) Spontaneous emission

$$A_2 \rightarrow A_1 + p \ ,$$

c.3) Stimulated emission

$$A_2 + p \rightarrow 2p + A_1 \ ,$$

where the photons involved in the last process have the same velocity $c\mathbf{\Omega}$, $\mathbf{\Omega} \in \mathcal{S}$, $\mathcal{S}$ the unit sphere in $\mathbb{R}^3$. We denote by $F(t, \mathbf{x}, \mathbf{\Omega})d\mathbf{x}d\mathbf{\Omega}$ the mean number of photons at time t that belong in the volume element $d\mathbf{x}$, with velocity in the direction $d\mathbf{\Omega}$. From now on, we shall consider the radiation intensity $I(t, \mathbf{x}, \mathbf{\Omega})$, related to F by

$$I(t, \mathbf{x}, \mathbf{\Omega}) = ch\nu F(t, \mathbf{x}, \mathbf{\Omega}) \ .$$

The events (c.1), (c.2) and (c.3) are respectively described by means of the so–called Einstein coefficients [RSM] α and β, which will be considered here as identification parameters of the model and therefore are assumed as known.

The evolution equation for $I(t, \mathbf{x}, \mathbf{\Omega})$ is given by

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mathbf{\Omega} \cdot \nabla I = h\nu[n_2(\alpha + \beta I) - n_1 \beta I] = h\nu J_p \ , \tag{2.1}$$

where n_1 and n_2 are, respectively, the number densities of particles A_1 and A_2.
For $f_1 = f_1(t, \mathbf{x}, \mathbf{v})$ and $f_2 = f_2(t, \mathbf{x}, \mathbf{v})$, the distribution functions of particles A_1 and A_2, respectively, and $\mathbf{v} \in \mathbb{R}^3$, particle's velocity, the kinetic Boltzmann–like equations for the two particle species A_1 and A_2 have the form

$$\frac{\partial f_k}{\partial t} + \mathbf{v} \cdot \nabla f_k = J_k^i + J_k^r + J_k^e = J_k \ , \tag{2.2}$$

where the $J_k^\ell = J_k^\ell(t, \mathbf{x}, \mathbf{v})$, $\ell = e, i, r$ are the collision integrals corresponding to the interaction classes (a), (b) and (c), respectively. In the Appendix we provide the expressions of the collisional integrals J_k^ℓ derived in paper [RSM].

In the same paper [RSM] thermodynamical equilibrium conditions have been derived for the equations (2.1) and (2.2). In particular, the elastic collision terms J_k^e vanish whenever the distribution functions are Maxwellians, that is

$$f_k(\mathbf{v}) = n_k \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left[-\frac{m(\mathbf{v} - \mathbf{u})^2}{2k_B T}\right] \ . \tag{2.3}$$

In order for the inelastic collision terms J_k^i to vanish, the the number densities must be linked by the following mass action law:

$$n_2/n_1 = \exp(-\Delta E/k_B T) \ . \tag{2.4}$$

Finally, the gas–radiation interaction terms J_k^r and J_p vanish as well if Planck's radiation law is fulfilled

$$I = \frac{\alpha/\beta}{\exp(h\nu/k_B T) - 1} \ . \tag{2.5}$$

3. Moment Equations and their Approximation A derivation of moment equations proceeds in a standard way; we multiply equations (2.2) by m, $m\mathbf{v}$, and $\frac{1}{2}mv^2$, respectively, then sum over k and finally integrate with respect to $d\mathbf{v}$. This yields,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{3.1}$$

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + \mathbb{P}) = \mathbf{0} \tag{3.2}$$

$$\frac{\partial}{\partial t}(e_{\text{th}} + \tfrac{1}{2}\rho u^2) + \nabla \cdot \{[(e_{\text{th}} + \tfrac{1}{2}\rho u^2)\mathbb{I} + \mathbb{P}] \cdot \mathbf{u} + \mathbf{q}_{\text{th}}\} = S^i \Delta E \ , \tag{3.3}$$

($\mathbb{I}$ is the unit tensor), where the macroscopic quantities (all functions of t and $\mathbf{x}$), related to the material gas only, are given by:

– mass density

$$\rho = m \sum_k \int_{\mathbb{R}^3} f_k(\mathbf{v}) d\mathbf{v} \ ,$$

– mean velocity

$$\mathbf{u} = \frac{m}{\rho} \sum_k \int_{\mathbb{R}^3} \mathbf{v} f_k(\mathbf{v}) d\mathbf{v} \ ,$$

– stress tensor

$$\mathbb{P} = m \sum_k \int_{\mathbb{R}^3} (\mathbf{v} - \mathbf{u}) \otimes (\mathbf{v} - \mathbf{u}) f_k(\mathbf{v}) d\mathbf{v} \ ,$$

– thermal energy density (linked to temperature T)

$$e_{\text{th}} = \frac{3\rho k_B T}{2m} = \tfrac{1}{2}\,\text{tr}(\mathbb{P}) \ ,$$

– heat flux

$$\mathbf{q}_{\text{th}} = \tfrac{1}{2} m \sum_k \int_{\mathbb{R}^3} (\mathbf{v} - \mathbf{u})(\mathbf{v} - \mathbf{u})^2 f_k(\mathbf{v}) d\mathbf{v} \ ,$$

– source of non–excited atoms, due to inelastic interactions only

$$\begin{aligned} S^i &= \int_{\mathbb{R}^3} J_1^i d\mathbf{v} = \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathcal{S}} g\{\sigma_{12}^{11}(g,\chi)[f_1(\mathbf{v})f_2(\mathbf{w}) - f_1(\mathbf{v}'_+)f_1(\mathbf{w}'_+)] + \\ &+ \sigma_{22}^{12}(g,\chi)[f_2(\mathbf{v})f_2(\mathbf{w}) - f_1(\mathbf{v}'_+)f_2(\mathbf{w}'_+)]\} d\mathbf{v} d\mathbf{w} d\mathbf{\Omega}' \ , \end{aligned}$$

where the cross sections σ_{12}^{11} and σ_{22}^{12} are defined in the Appendix.

Equations (3.1) and (3.2) represent, respectively, mass and momentum conservation for the material gas. Equation (3.3) is the kinetic energy balance for atoms.

Next, by integrating the equation of photon transport (2.1) with respect to $d\mathbf{\Omega}$, we obtain

$$\frac{\partial e_{\text{r}}}{\partial t} + \nabla \cdot \mathbf{j}_{\text{r}} = h\nu[n_2(4\pi\alpha + c\beta e_{\text{r}}) - n_1 c\beta e_{\text{r}}] \ , \tag{3.4}$$

where

$$e_{\text{r}} = \frac{\mathcal{I}}{c} \quad \text{with} \quad \mathcal{I} = \int_{\mathcal{S}} I(\mathbf{\Omega}) d\mathbf{\Omega}$$

is the radiation energy density, and

$$\mathbf{j}_{\mathrm{r}} = \int_{\mathcal{S}} \mathbf{\Omega} I(\mathbf{\Omega}) d\mathbf{\Omega}$$

is the radiative flux.

An additional independent balance equation can be obtained by considering Eq.(2.1) together with Eqs.(2.2). Indeed, after multiplying these equations by $\frac{1}{2}mv^2 + E_k$, summing them over k, then integrating with respect to $d\mathbf{v}$, and finally by adding this result to Eq.(3.4), we obtain the total energy conservation equation for the whole system:

$$\frac{\partial}{\partial t}(e_A + e_{\mathrm{r}}) + \nabla \cdot [(e_A \mathbb{I} + \mathbb{P}) \cdot \mathbf{u} + \mathbf{q}_{\mathrm{th}} + \mathbf{q}_{\mathrm{exc}} + \mathbf{j}_{\mathrm{r}}] = 0 \ , \tag{3.5}$$

where

$$e_A = \tfrac{1}{2}\rho u^2 + e_{\mathrm{th}} + e_{\mathrm{exc}}$$

is the total energy for atoms, with

$$e_{\mathrm{exc}} = \sum_k E_k \int_{\mathbb{R}^3} f_k(\mathbf{v}) d\mathbf{v} = E_1 n_1 + E_2 n_2 \ ,$$

as excitation energy density, and where

$$\mathbf{q}_{\mathrm{exc}} = \sum_k E_k \int_{\mathbb{R}^3} (\mathbf{v} - \mathbf{u}) f_k(\mathbf{v}) d\mathbf{v} \ ,$$

represents the diffusion current of excitation energy. Let us observe that all the macroscopic quantities appearing in Eqs.(3.4–3.5) are functions of t and $\mathbf{x}$, as well.

The equations (3.1), (3.2), and (3.5) constitute the macroscopic conservation equations of the system. We note that equations (3.1–3.3, 3.5), together with Eq.(2.1), are exact, however, they do not form a closed system of equations.

In order to close the above system, we observe that when the characteristic relaxation time of elastic collision processes is much smaller compared with the ones relevant to inelastic and gas–radiation interaction processes, one may adopt the standard zero–order Chapman–Enskog approximation.

In this case one can calculate $\mathbf{q}_{\mathrm{th}}$, $\mathbf{q}_{\mathrm{exc}}$, $\mathbf{u}_k$ and S^i by means of Maxwellians (2.3), namely

$$\mathbf{q}_{\mathrm{th}} = \mathbf{q}_{\mathrm{exc}} = \mathbf{0} \ , \qquad \mathbf{u}_k = \mathbf{u} \ , \tag{3.7}$$

$$S^i = [n_1 \gamma_{12}(T) + n_2 \gamma_{22}(T)][n_2 - n_1 \exp(-\Delta E / k_B T)] \ , \tag{3.8}$$

where

$$\begin{bmatrix} \gamma_{12}(T) \\ \\ \gamma_{22}(T) \end{bmatrix} = \left(\frac{m}{2\pi k_B T} \right)^3 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathcal{S}} \exp\left(-\frac{m}{2k_B T}(v^2 + w^2) \right) g \begin{bmatrix} \sigma_{12}^{11}(g,\chi) \\ \\ \sigma_{22}^{12}(g,\chi) \end{bmatrix} d\mathbf{v} d\mathbf{w} d\mathbf{\Theta}' \ .$$

Using (3.7) and (3.8) in one dimensional case, i.e., with

$$\nabla = \mathbf{i}\, \frac{\partial}{\partial x} \ , \qquad \mathbf{u} = u\mathbf{i} \ , \qquad I = I(t, x, \mu) \ ,$$

$\mathbf{i}$ being the unit vector in the direction x and $\mu = \mathbf{\Omega} \cdot \mathbf{i} \in [-1, 1]$. equations (3.1, 3.2, 3.3, 3.6b) have the form

$$\frac{\partial}{\partial t}\Big(n_1 + n_2 \Big) + \frac{\partial}{\partial x}\Big[u(n_1 + n_2) \Big] = 0 \tag{3.9}$$

$$\frac{\partial}{\partial t}\Big[u(n_1+n_2)\Big]+\frac{\partial}{\partial x}\Big[u^2(n_1+n_2)\Big]+\frac{1}{m}\frac{\partial p}{\partial x}=0 \tag{3.10}$$

$$\begin{aligned}\frac{\partial}{\partial t}\Big[(n_1+n_2)(\tfrac{3}{2}k_BT+\tfrac{1}{2}mu^2)\Big]+\frac{\partial}{\partial x}\Big[u(n_1+n_2)(\tfrac{3}{2}k_BT+\tfrac{1}{2}mu^2)+p\Big]=\\ =\Delta E\Big[n_2-n_1\exp\left(-\frac{\Delta E}{k_BT}\right)\Big][n_1\gamma_{12}(T)+n_2\gamma_{22}(T)]\end{aligned} \tag{3.11}$$

$$\begin{aligned}\frac{\partial n_1}{\partial t}+\frac{\partial}{\partial x}\Big(un_1\Big)=-[n_1\gamma_{12}(T)+n_2\gamma_{22}(T)]\times\\ \times[n_2-n_1\exp(-\Delta E/k_BT)]+n_1\beta\mathcal{I}-n_2(4\pi\alpha+\beta\mathcal{I})\ .\end{aligned} \tag{3.12}$$

where $p=\mathbf{i}\cdot\mathbb{P}\cdot\mathbf{i}$.

4 The Milne–Chandrasekhar Problem

The problem has its origin in the experiment performed by Hayner in 1925 (see [MIL] and references therein). Suppose that a slab filled with a monoatomic gas (say mercury vapour) is illuminated from one side by light (say from a mercury arc) in resonance with these atoms for a length of time sufficient to establish a steady state. Then we suddenly cut off the slab from the source of illumination. The problem consists in studying two transients cases: before and after the cut off.

The latter case was studied first by Milne [MIL] in 1926 and then by Chandrasekhar [CHA] in 1949. Both, Milne and Chandrasekhar, considered a material gas at a fixed density. Furthermore, they did not consider any mechanism of exchange between the internal energy levels of atoms. Finally, the equations they actually solved were linearized. Here, the same problem is treated in full generality. In other words,

i) we keep all the sources of nonlinearity,
ii) we consider the effects of inelastic interactions,
iii) we do not neglect $(1/c)(\partial I/\partial t)$.

In the case of a gas in a slab ($-a\le x\le a$) and a fixed density n, we assume

a) For $t\in(-\infty,0)$ the gas at a known number density n is in absolute equilibrium at a temperature T_0, together with the radiation field.
b) The boundaries of the slab are perfectly reflecting mirrors, so that also the radiation field, at temperature T_0, is given by the Planck law stated by Eq.(2.5).
c) At $t=0^+$ the mirrors are removed and the gas in the slab is subjected to zero radiation at the left boundary $x=-a$ and to a known radiation $I^*(\mu)$ at the right one $x=a$.
d) The boundary conditions for the observable u are such that $u(\pm a)=0$.

Under the above assumptions the system of equations (3.9–3.12) can be written in the form

$$\frac{\partial T}{\partial t}=\frac{2h\nu}{3nk_B}[n_1\gamma_{12}(T)+n_2\gamma_{22}(T)][n_2-n_1\exp(-h\nu/k_BT)]\ , \tag{4.1}$$

$$\frac{\partial n_2}{\partial t}=[n_1\gamma_{12}(T)+n_2\gamma_{22}(T)][n_1\exp(-h\nu/k_BT)-n_2]+\beta\mathcal{I}(n_1-n_2)-4\pi\alpha n_2\ , \tag{4.2}$$

$$\frac{1}{c}\frac{\partial I}{\partial t}+\mu\frac{\partial I}{\partial x}=h\nu[\alpha n_2-\beta I(n_1-n_2)]\ , \tag{4.3}$$

where $n_1=n-n_2$, with the initial data

$$\forall x\in[-a,a]:\qquad\begin{cases}I(0,x,\mu)=\dfrac{\alpha/\beta}{\exp(h\nu/k_BT_0)-1}\quad\forall\mu\in[-1,1]\\[2ex] n_2(0,x)=\dfrac{n}{\exp(h\nu/k_BT_0)+1}\\[2ex] T(0,x)=T_0\end{cases} \tag{4.4}$$

and with the boundary conditions

$$\forall t \in (0, t_M): \qquad \begin{cases} I(t,-a,\mu) = 0 \ , \quad \forall \mu > 0 \\ I(t,a,\mu) = I^*(\mu) \ , \quad \forall \mu < 0 \ . \end{cases} \tag{4.5}$$

Equations (4.2) and (4.3) reduce themselves to the Chandrasekhar model [CHA] when we set $\gamma_{12} = \gamma_{22} = 0$, neglect $(1/c)(\partial I/\partial t)$ with respect to $\mu(\partial I/\partial x)$, and, under the assumption that n_2 is much smaller than n_1, approximate $n_1 - n_2$ by n_1, where n_1 is treated as a constant.

5 Mathematical analysis of the system: existence and uniqueness results

In a dimensioneless form the system of equations (4.1-4.3) can be written as

$$\begin{aligned}
\frac{\partial T}{\partial t} &= \frac{2}{3}\rho \left\{ \gamma_{12}(T)(1-n_2)n_2 + \gamma_{22}(T)n_2^2 - \left[\gamma_{12}(T)(1-n_2)^2 + \gamma_{22}(T)(1-n_2)n_2\right] \exp\left(-\frac{1}{T}\right) \right\}, \\
\frac{\partial n_2}{\partial t} &= \rho \left\{ \left[\gamma_{12}(T)(1-n_2)^2 + \gamma_{22}(T)(1-n_2)n_2\right] \exp\left(-\frac{1}{T}\right) - \gamma_{12}(T)(1-n_2)n_2 - \gamma_{22}(T)n_2^2 \right\} \\
&\quad + \eta\left[\mathcal{I} - n_2(4\pi\lambda + 2\mathcal{I})\right], \\
\frac{\partial I}{\partial t} &+ \mu\frac{\partial I}{\partial x} = \lambda n_2 + (2n_2 - 1)I,
\end{aligned} \tag{5.1 - 5.3}$$

together with the initial conditions

$$T(0,x) = T_0(x), \quad n_2(0,x) = n_{20}(x), \quad I(0,x,\mu) = I_0(x,\mu) \quad \forall x \in [-a,a], \quad \forall \mu \in [-1,1]$$

and the boundary conditions

$$\begin{cases} I(t,-a,\mu) = 0 & \forall \mu > 0, \\ I(t,a,\mu) \ \ = I^*(\mu) & \forall \mu < 0 \ . \end{cases}$$

Here, as before, $\mathcal{I}(t,x) = 2\pi \int_{-1}^{1} I(t,x,\mu)\, d\mu$, and $\gamma_{12}(T)$, $\gamma_{22}(T) \approx T^{\frac{p-5}{2(p-2)}}$ when inelastic cross-sections, $\sigma_{12}^{11}(g,\chi)$, $\sigma_{22}^{12}(g,\chi)$, are given according to the inverse power potential, with exponent $p-1$, and with the standard angular cutoff (see, for example, [GRA]). We recall that so called Maxwell's molecules correspond to $p = 5$, when $\gamma_{12}(T)$, $\gamma_{22}(T)$ do not depend on the temperature T. In the present work we consider only this case, i.e., γ_{12}, γ_{22} are given positive constants. Furthermore, in the system above, the densities n_2 and n_1 are normalized to give $n_1(t,x) + n_2(t,x) = 1$, for $t \geq 0$ and $x \in [-a,a]$, and ρ, η, and λ are given positive constants. Finally, while dealing with the mathematical analysis of the problem, we treat more general initial conditions as compared to (4.4).

We look for physically meaningful solutions for (5.1-5.3); this means, among other things, that $T(t,x)$, $n_2(t,x)$, $I(t,x,\mu) \geq 0$, for $t > 0$, $x \in [-a,a]$, and $\mu \in [-1,1]$, if the initial conditions T_0, n_{20}, I_0 and the boundary condition I^* (at $x = a$) are also nonnegative. In addition, the density, $n_2(t,x)$, should satisfy $0 \leq n_2(t,x) \leq 1$ for $t \geq 0$ and $x \in [-a,a]$.

The proof of the existence result for (5.1-5.3) is divided into several steps:

1. Derivation of a priori estimation for the problem.
2. Solution of an approximated problem.
3. Passage to the limit with the sequence of approximated solutions and the proof that the limit is a solution to the original problem.

5.1 A priori estimations

In this section we state several a priori estimations for the system (5.1-5.3) that will be used later in the section. The proofs of these estimations as well as some other theorems are quite technical and are postoponed to the forthcoming paper of the same authors.

Proposition 5.1. *Suppose that the system (5.1-5.3) has smooth and nonnegative solutions $T(t,x)$, $n_2(t,x)$, $I(t,x,\mu)$, and $0 \le n_2(0,x) \le 1$, for $x \in [-a,a]$. Then*

$$0 \le n_2(t,x) \le 1, \qquad t > 0, \quad x \in [-a,a]. \tag{5.4}$$

Proof: First, we prove the inequality $||n_2||_{L^\infty} = \sup_{t,\,x} |n_2(t,x)| \le 3/2$ and then that actually $||n_2||_{L^\infty} \le 1$. We observe that under the assumptions of the proposition the equation

$$\begin{aligned}\frac{\partial n_2}{\partial t} = \rho \Big\{ \big[\gamma_{12}|1-n_2|^2 + \gamma_{22}|1-n_2|n_2\big] \exp\left(-\frac{1}{T}\right) - \gamma_{12}|1-n_2|n_2 - \gamma_{22}n_2^2 \Big\} \\ + \eta\left[\mathcal{I} - n_2(4\pi\lambda + 2\mathcal{I})\right]\end{aligned} \tag{$*$}$$

is equivalent to the equation (5.2), if $||n_2||_{L^\infty} \le 1$. On the other hand, for $t \ge 0$, equation (*) can be written as

$$\begin{aligned} n_2(t,x) = n_2(0,x)\exp\left(-\int_0^t \Lambda(s)\right)\, ds + \int_0^t \Big\{ |1-n_2(s)| \Big[\rho\gamma_{12}|1-n_2(s)| \\ + \rho\gamma_{22}n_2\Big] \exp\left(-\frac{1}{T}\right) + \eta\mathcal{I}(s)\Big\} \exp\left(-\int_s^t \Lambda(r)dr\right) ds, \end{aligned} \tag{$**$}$$

where $\Lambda(s) = \rho\gamma_{12}|1-n_2(s)| + \rho\gamma_{22}n_2(s) + 4\pi\lambda\eta + 2\eta\mathcal{I}(s)$. Therefore, since $0 \le n_2(0,x) \le 1$,

$$\begin{aligned} n_2(t,x) \le{}& \frac{1}{2}\exp\left(-\int_0^t (4\pi\lambda\eta + 2\eta\mathcal{I}(s))\, ds\right) + \frac{1}{2}\left\{1 - \exp\left(-\int_0^t (4\pi\lambda\eta + 2\eta\mathcal{I}(s))\, ds\right)\right\} \\ &+ \frac{1}{2}\exp\left(-\int_0^t [\rho\gamma_{12}|1-n_2(s)| + \rho\gamma_2 n_2(s)]\, ds\right) \\ &+ ||1-n_2||_{L^\infty}\left\{1 - \exp\left(-\int_0^t [\rho\gamma_{12}|1-n_2(s)| + \rho\gamma_2 n_2(s)]\, ds\right)\right\}, \end{aligned}$$

and we obtain the inequalities (since $n_2(t,x) \ge 0$),

$$||n_2||_{L^\infty} \le \frac{1}{2} + \max\left(\frac{1}{2},\, ||1-n_2||_{L^\infty}\right) \le \frac{1}{2} + \max\left\{\frac{1}{2}, \max\Big(1, ||n_2||_{L^\infty} - 1\Big)\right\}.$$

The last inequality gives $||n_2||_{L^\infty} \le 3/2$. This also means that $||1-n_2||_{L^\infty} \le 1$ and by using this fact, identity (**) yields the inequalities

$$\begin{aligned} n_2(t,x) \le{}& \exp(-\int_0^t \Lambda(s))ds \\ &+ \int_0^t \Big\{\rho\gamma_{12}|1-n_2(s)| + \rho\gamma_{22}n_2(s) + 4\pi\lambda\eta + 2\eta\mathcal{I}(s)\Big\} \exp\Big(-\int_s^t \Lambda(r)dr\Big) ds \\ ={}& \exp(-\int_0^t \Lambda(s))\, ds + \int_0^t \left\{\frac{d}{ds}\exp(-\int_s^t \Lambda(r)\, dr)\right\}\, ds \\ \le{}& \exp(-\int_0^t \Lambda(s))\, ds + \left\{1 - \exp(-\int_0^t \Lambda(s)\, ds)\right\} = 1. \end{aligned}$$

We note that the proof of the inequality (5.4) does not depend on a choice of nonnegative constants $\rho, \eta, \lambda, \gamma_{12}, \gamma_{22}$.

In our numerical simulations we will often deal with the cases when both the initial value, $n_{20}(x)$, and the solution, $n_2(t,x)$, should satisfy $0 \le n_{20}(x), n_2(t,x) \le \frac{1}{2}$, for $t > 0$ and $x \in [-a,a]$. Under certain conditions on the parameters $\rho, \eta, \lambda, \gamma_{12}, \gamma_{22}$ the following result holds:

Proposition 5.2. *Suppose that the system (5.1-5.3) has smooth and nonnegative solutions $T(t,x)$, $n_2(t,x)$, $I(t,x,\mu)$, and $0 \le n_2(0,x) \le d$, for $x \in [-a,a]$, where $\frac{1}{2} \le d \le 1$. Then*

$$0 \le n_2(t,x) \le d, \qquad t>0, \quad x\in[-a,a], \tag{5.5}$$

if

$$\left(\frac{1}{d}-1\right)\rho\max\left(\gamma_{12},\gamma_{22}\right) \le 4\pi\lambda\eta. \tag{5.6}$$

Proof: By Proposition 5.1, $||n_2||_{L^\infty} \le 1$, as well as $||1-n_2||_{L^\infty} \le 1$. Now, identity (**) yields

$$\begin{aligned} n_2(t,x) &\le d\exp(-\int_0^t \Lambda(s))ds \\ &+ d\int_0^t \frac{1}{d}\Big\{\rho\gamma_{12}|1-n_2(s)| + \rho\gamma_{22}n_2(s) + \eta\mathcal{I}(s)\Big\}\exp\Big(-\int_s^t \Lambda(r)dr\Big)ds, \end{aligned}$$

where as before, $\Lambda(s) = \rho\gamma_{12}|1-n_2(s)| + \rho\gamma_{22}n_2(s) + 4\pi\lambda\eta + 2\eta\mathcal{I}(s)$. The proof will be complete if we show that

$$\frac{1}{d}\Big\{\rho\gamma_{12}|1-n_2(s)| + \rho\gamma_{22}n_2(s) + \eta\mathcal{I}(s)\Big\} \le \rho\gamma_{12}|1-n_2(s)| + \rho\gamma_{22}n_2(s) + 4\pi\lambda\eta + 2\eta\mathcal{I}(s).$$

However, by (5.6),

$$\left(\frac{1}{d}-1\right)\{\rho\gamma_{12}|1-n_2(s)| + \rho\gamma_{22}n_2(s)\} \le \left(\frac{1}{d}-1\right)\rho\max\left(\gamma_{12},\gamma_{22}\right) \le 4\pi\lambda\eta + \left(2-\frac{1}{d}\right)\eta\mathcal{I},$$

since by the assumption, $(2-1/d)\ge 0$. This completes the proof.

The next two estimations together with Propositions 5.1 (or 5.2) give bounds on the temperature field $T(t,x)$ and the photon intensity $I(t,x,\mu)$.

Proposition 5.3. *Suppose that the system (5.1-5.3) has smooth and nonnegative solutions $T(t,x)$, $n_2(t,x)$, $I(t,x,\mu)$, and $n_2(0,x), T(0,x) \ge 0$. Then*

$$n_2(t,x) + \frac{3}{2}T(t,x) \le e\cdot\max\left\{\frac{1}{2}, \sup_{x\in[-a,a]}\left(n_2(0,x)+\frac{3}{2}T(0,x)\right)\right\}, t>0, x\in[-a,a]. \tag{5.7}$$

Proof: By adding equation (5.1) to the equation (5.2) multiplied by 3/2, and then by adding and substracting the expression $(3/2)T\left(4\pi\lambda\eta + 2\eta\mathcal{I}\right)$, we obtain the identity

$$\begin{aligned} n_2(t,x) + \frac{3}{2}T(t,x) &= \left(n_2(0,x) + \frac{3}{2}T(0,x)\right)\exp\left\{-\int_0^t (4\pi\lambda\eta + 2\eta\mathcal{I}(s,x))\,ds\right\} \\ &+ \int_0^t \left\{\left[\eta\mathcal{I} + \frac{3}{2}T(s,x)\left(4\pi\lambda\eta + 2\eta\mathcal{I}(s,x)\right)\right]\exp\left[-\int_s^t (4\pi\lambda\eta + 2\eta\mathcal{I}(r,x))dr\right]\right\}\,ds. \end{aligned}$$

Now, we have

$$\begin{aligned} n_2(t,x) + \frac{3}{2}T(t,x) &\le \left(n_2(0,x) + \frac{3}{2}T(0,x)\right)\exp\left\{-\int_0^t (4\pi\lambda\eta + 2\eta\mathcal{I}(s,x))\right\} \\ &+ \frac{1}{2}\int_0^t \left[4\pi\lambda\eta + 2\eta\mathcal{I}(s,x)\right]\exp\left\{-\int_s^t (4\pi\lambda\eta + 2\eta\mathcal{I}(r,x))\,dr\right\} \\ &+ \int_0^t \frac{3}{2}T(s,x)\frac{d}{ds}\left\{\exp\left[-\int_s^t (4\pi\lambda\eta + 2\eta\mathcal{I}(r,x))dr\right]\right\}\,ds \\ &= \left(n_2(0,x) + \frac{3}{2}T(0,x)\right)\exp\left\{-\int_0^t (4\pi\lambda\eta + 2\eta\mathcal{I}(s,x))\right\} \\ &+ \int_0^t \left[\frac{1}{2} + \frac{3}{2}T(s,x)\right]\frac{d}{ds}\left\{\exp\left[-\int_s^t (4\pi\lambda\eta + 2\eta\mathcal{I}(r,x))dr\right]\right\}\,ds, \end{aligned}$$

and using nonengativity of n_2,

$$n_2(t,x)+\frac{3}{2}T(t,x)\le \max\left\{\frac{1}{2},\sup_{x\in[-a,a]}\left(n_2(0,x)+\frac{3}{2}T(0,x)\right)\right\}$$
$$+\int_0^t\left[n_2(s,x)+\frac{3}{2}T(s,x)\right]\frac{d}{ds}\left\{\exp\left[-\int_s^t(4\pi\lambda\eta+2\eta\mathcal{I}(r,x))dr\right]\right\}ds.$$

Next, we need the following generalization of Gronwall's inequality [MPF, Theorem 1, p. 358]:

Lemma [H. Movljankulov and A. Filatov]. *Let $u(t)$ be real, continuous, and nonnegative such that for $t\ge t_0$*

$$u(t)\le c+\int_{t_0}^t k(t,s)u(s)\,ds,\qquad c>0,$$

where $k(t,s)$ is a continuously differentiable function in t and continuous in s with $k(t,s)\ge 0$ for $t\ge s\ge t_0$. Then

$$u(t)\le c\exp\left\{\int_{t_0}^t\left[k(s,s)+\int_{t_0}^s\frac{\partial k}{\partial s}(s,r)\,dr\right]ds\right\}.$$

Now, for each fixed x, we apply the above Lemma with $t_0=0$ and

$$k(t,s)=\frac{d}{ds}\exp\left[-\int_s^t(4\pi\lambda\eta+2\eta\mathcal{I}(r,x))dr\right]$$

to obtain

$$\exp\left\{\int_{t_0}^t\left[k(s,s)+\int_{t_0}^s\frac{\partial k}{\partial s}(s,r)\,dr\right]ds\right\}=\exp\left[1-\exp\left(-\int_0^t(4\pi\lambda\eta+2\eta\mathcal{I}(r,x))\;dr\right)\right].$$

This completes the proof.

Proposition 5.4. *Assume that $0\le n_2(0,x)\le 1$ and $0\le T(0,x),I(0,x,\mu)\le C$, for some $C>0$, and for $x\in[-a,a]$, $\mu\in[-1,1]$. In addition, assume that $0\le I^*(\mu)\le C$, for $-1\le\mu\le 0$. Finally, suppose that the system (5.1-5.3) has smooth and nonnegative solutions $T(t,x),n_2(t,x),I(t,x,\mu)$ on the time interval $[0,\tau]$. Then there is $C_\tau>0$ such that*

$$0\le I(t,x,\mu)\le C_\tau,\qquad t\in[0,\tau],\quad x\in[-a,a],\quad \mu\in[-1,1]. \tag{5.8}$$

Proof: By adding a suitable source term, $\widehat{I}^*(t,x,\mu)$, (see, for example, [GMP, p. 413-416]) to the right hand side of (5.3), we can transform the nonhomogeneous boundary condition into the homogeneous boundary condition, and write $I(t,x,\mu)$ in the form

$$I(t,x,\mu)=(S(t)I_0)(x,\mu)+\int_0^t\left(S(t-s)\left[\lambda n_2(s)+(2n_2(s)-1)I(s)+\widehat{I}^*(s)\right]\right)(x,\mu)\,ds,$$

where $S(t)$ is the evolution operator associated with the operator $-\mu\frac{\partial I}{\partial x}$ and the corresponding homogeneous boundary condition. The boundedness of I^* implies the boundedness of $\widehat{I}^*(t,x,\mu)$ for $t\in[0,\tau]$, $x\in[-a,a]$, and $\mu\in[-1,1]$. Finally, since $1\le n_2(t,x)\le 1$, Gronwall's lemmma completes the proof.

We end this section with the following energy estimation:

Proposition 5.5. *Suppose that the system (5.1-5.3) has smooth and nonnegative solutions $T(t,x)$, $n_2(t,x)$, $I(t,x,\mu)$, and $I^*(\mu)\geq 0$, for $\mu\in[-1,0]$. Then, for $t\geq 0$,*

$$\frac{\partial}{\partial t}\int_{-a}^{a}\left[n_2(t,x)+\frac{3}{2}T(t,x)+\eta\mathcal{I}(t,x)\right]\,dx\leq -\eta\int_{-1}^{0}\mu I^*(\mu)\,d\mu. \tag{5.9}$$

Proof: The following energy identity together with nonnegativity of $I(t,x,\mu)$ completes the proof:

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{-a}^{a}\left[n_2(t,x)+\frac{3}{2}T(t,x)+\eta\mathcal{I}(t,x)\right]\,dx\\&\qquad=\eta\int_{-1}^{0}\mu I(a,\mu)\,d\mu-\eta\int_{0}^{1}\mu I(a,\mu)\,d\mu-\eta\int_{-1}^{0}\mu I^*(\mu)\,d\mu.\end{aligned}$$

5.2 The approximated problem

The main idea is to construct a suitable approximated (or modified) problem to the original system (5.1-5.3) that retains the a priori estimations of section 5.1 and can be solved relatively easy.

For $k\geq 1$, we consider the following sequence of approximated problems for $t\in[0,\tau]$, $\tau>0$:

$$\begin{aligned}\frac{\partial T_k}{\partial t}&=\frac{2}{3}\rho\left\{\gamma_{12}(1-n_2^k)n_2^k+\gamma_{22}(n_2^k)^2-\left[\gamma_{12}(1-n_2^k)^2+\gamma_{22}(1-n_2^k)n_2^k\right]\exp\left(-\frac{1}{T_k}\right)\right\},\\ \frac{\partial n_2^k}{\partial t}&=\rho\left\{\left[\gamma_{12}(1-n_2^k)^2+\gamma_{22}(1-n_2^k)n_2^k\right]\exp\left(-\frac{1}{T_k}\right)-\gamma_{12}(1-n_2^k)n_2^k-\gamma_{22}(n_2^k)^2\right\}\\&\quad+\eta\left[\frac{\mathcal{I}_k}{1+\frac{1}{k}\mathcal{I}_k}-n_2^k\left(4\pi\lambda+2\frac{\mathcal{I}_k}{1+\frac{1}{k}\mathcal{I}_k}\right)\right],\\ \frac{\partial I_k}{\partial t}+\mu\frac{\partial I_k}{\partial x}&=\lambda n_2^k+2n_2^k\frac{I_k}{1+\frac{1}{k}\mathcal{I}_k}-I_k,\end{aligned}\tag{5.10 – 5.12}$$

together with the initial conditions

$$T_k(0,x)=T_0(x),\qquad n_2^k(0,x)=n_{20}(x),\qquad I_k(0,x,\mu)=I_0(x,\mu)\quad\forall x\in[-a,a],$$

and the boundary conditions

$$\begin{cases}I_k(t,-a,\mu)=0 & \forall\mu>0,\\ I_k(t,a,\mu)\;=I^*(\mu) & \forall\mu<0\;.\end{cases}$$

In addition, we set $n_2^0(t,x)=n_{20}(x)$.

It is well known (see, for example, [GMP, pp. 413-416] that the above nonhomogeneous boundary conditions, with $0\leq I^*\in L^\infty(-1,0)$, can be transformed into the homogeneous boundary condition by adding a nonhomogeneous (source) term, $\widehat{I}^*(t,x,\mu)$, to the right hand side of equation (5.12). In addition, $\widehat{I}^*(t,x,\mu)$ can be constructed in such a way that $0\leq\widehat{I}^*\in L^\infty((0,T)\times(-a,a)\times(-1,1))$. From now on this system will be denoted as $(5.10-5.12)_0$.

For each fixed k, we consider the system $(5.10-5.12)_0$ as a semilinear problem in the Banach space $X=L^1(-a,a)\times L^1(-a,a)\times L^1((-a,a)\times(-1,1))$. In order to avoid cumbersome notation, the index k, appearing in T, n_2, and I, will be dropped from now on. Next, let X^+ be a subset of X consisting of elements $u=(T,n_2,I)$ such that $T\in L^1(-a,a)$, $n_2\in L^1(-a,a)$, and $I\in L^1((-a,a)\times(-1,1))$ are nonnegative (almost everywhere) functions on $(-a,a)$, $(-a,a)$, and $((-a,a)\times(-1,1))$, respectively.

Let $D\subset X$ be defined as

$$D=\{u\in X^+:\,0\leq n_2(x)\leq 1\ \text{ almost everywhere in } x\in(-a,a)\}\tag{5.13}$$

and $C([0,\tau],X)$ be the Banach space of continuous functions from $[0,\tau]\to X$. For $k\geq 1$, the system $(5.10-5.12)_0$ is equivalent to the following abstract semilinear problem on the closed set $D\subset X$:

$$u'(t) = Au(t) + B_k(t,u(t)) \quad \text{for } t\in(0,\tau], \quad u(0)=u_0\in D, \tag{5.14}$$

where for $u=(T,n_2,I)$,

$$Au = \begin{pmatrix} 0 \\ 0 \\ -\mu\frac{\partial I}{\partial x} - I \end{pmatrix}, \tag{5.15}$$

with the domain $\mathcal{D}(A)$ defined by

$$\mathcal{D}(A) = \{u\in X:\ \mu\frac{\partial I}{\partial x}\in L^1((-a,a)\times(-1,1)) \text{ plus the homogeneous boundary condition}\},$$

and, for $(t,u)\in[0,\tau]\times D$,

$$B_k(t,u) = \begin{pmatrix} \frac{2}{3}\rho\left\{\gamma_{12}(1-n_2)n_2+\gamma_{22}(n_2)^2-\left[\gamma_{12}(1-n_2)^2+\gamma_{22}(1-n_2)n_2\right]\exp\left(-\frac{1}{T}\right)\right\} \\ \rho\left\{\left[\gamma_{12}(1-n_2)^2+\gamma_{22}(1-n_2)n_2\right]\exp\left(-\frac{1}{T}\right)-\gamma_{12}(1-n_2)n_2-\gamma_{22}(n_2)^2\right\}+\Gamma \\ \lambda n_2 + 2n_2\frac{I}{1+\frac{1}{k}\mathcal{I}} + \widehat{I}^*(t) \end{pmatrix},$$

with

$$\Gamma = \eta\left[\frac{\mathcal{I}}{1+\frac{1}{k}\mathcal{I}} - n_2\left(4\pi\lambda + 2\frac{\mathcal{I}}{1+\frac{1}{k}\mathcal{I}}\right)\right],$$

$\mathcal{I}(x) = 2\pi\int_1^1 I(x,\mu)\,d\mu$, and $\exp\left(-\frac{1}{T}\right)=0$, for $T=0$.

We remark that the homogeneous boundary condition appearing in the domain of the linear operator A is understood in the sense of trace class (see, for example, [CES] or [GMP]).

The existence and uniqueness result is based on the following properties of the initial value problem (5.14):

(1) The linear operator A generates a strongly continuous semigroup $U(t)$ in X such that $U(t): D\to D$, for $t\geq 0$.

(2) The nonlinear operator $B(t,u)$ is a continuous mapping from $[0,\tau]\times D$ into X that is also Lipschitz continuous on D, i.e.,

$$||B_k(t,u_1)-B_k(t,u_2)||_X \leq L(k)||u_1-u_2||_X \text{ for all } (t,u_1),\ (t,u_2)\in[0,\tau]\times D,$$

for some $L(k)>0$. Here, $||\cdot||_X$ denotes any equivalent norm on X.

(3) The set $D\subset X$ is invariant for the evolution problem (5.14).

Property (1) is the standard result that can be found in [GMP, Theorem 2.2, p. 410]. Property (2) follows after somewhat lenghtly but easy computations; here, the crucial is the fact that $n_2\in L^\infty(-a,a)$. Finally, property (3) follows from the following observation (see, [MAR, Theorem 2.1, p. 335]):

Proposition 5.6. *For a fixed $k\geq 1$, and all $(t,u)\in[0,\tau]\times D$*

$$\lim_{h\to 0^+}\frac{dist(u+hB_k(t,u);D)}{h} = 0, \tag{5.16}$$

where $dist(v,D)=\inf_{u\in D}||v-u||_X$ is the distance of an element $v\in X$ from the set D.

The limit in (5.16) is a different formulation of the first a priori estimation (Proposition 5.1) for the approximated problem (5.10-5.12). It also implies that the evolution problem (5.15) generates nonnegative solutions if we start with nonnegative initial values.

We have (see, Theorem 2.1, p. 335, and Remark 2.1, p. 337 of [MAR])

Theorem 5.7. *For each $u_0 \in D$ there is a unique continuous function $u : [0,\tau] \to D$ that satisfies*

$$u(t) = U(t)u_0 + \int_0^t U(t-s)B_k(s,u(s))\,ds \qquad \text{for all } t \in [0,\tau], \tag{5.17}$$

where the integral in (5.17) is the Riemann integral in X.

A function u that satisfies (5.17) is said to be a mild solution to (5.14) on $[0,\tau]$.

We note that $U(t)$ has the following representation for $u = (T, n_2, I)$:

$$U(t)u = \begin{pmatrix} T \\ n_2 \\ U_0(t)I \end{pmatrix}, \tag{5.18}$$

where $U_0(t)$ is a strongly continuous semigroup in $L^1((-a,a)\times(-1,1))$, generated by the operator $-\mu\frac{\partial I}{\partial x} - I$ with the homogeneous boundary condition (see, [GMP, Theorem 2.2, p. 410]).

Next, it follows that, for each $k \geq 1$, and $u_0 \in D \cap \mathcal{D}(A)$, Propositions 5.3-5.4 still hold for $u(t)$ given by (5.17). On the other hand, for $u_0 \in D$, we choose a sequence $\{u_{0k}\} \subset X$ such that $u_{0k} \to u_0$ in X, as $k \to \infty$, with $u_{0k} = (T_{0k}, n_{20}^k, I_{0k}) \in D \cap \mathcal{D}(A)$, and $T_{0k} \in L^\infty(-a,a)$, $I_{0k} \in L^\infty((-a,a)\times(-1,1))$ (uniformly in $k \geq 1$). Then again, by Propositions 5.3-5.4, the solution (now indexed by k), $u_k(t) = (T_k(t), n_2^k(t), I_k(t))$, satisfies $T_k(\cdot), n_2^k(\cdot) \in L^\infty([0,\tau]\times(-a,a))$ and $I_k(\cdot) \in L^\infty([0,\tau]\times(-a,a)\times(-1,1))$, uniformly in $k \geq 1$.

These estimations alone imply that the sequence of solutions $u_k(\cdot) = (T_k(\cdot), n_2^k(\cdot), I_k(\cdot))$, $k = 1,2,\dots$ is weakly compact in $L^1([0,\tau];X))$. Here, the weak compactness in $L^1([0,\tau];X))$ means that the sequences $\{T_k(\cdot)\}$, $\{n_2^k(\cdot)\}$ are weakly compact in $L^1([0,\tau]\times(-a,a))$, and the sequence $\{I_k(\cdot)\}$ is weakly compact in $L^1([0,\tau]\times(-a,a)\times(-1,1))$.

This type of weak convergence (after passing to a subsequence) is not enough to pass to the limit (when $k \to \infty$) in nonlinear terms of (5.10-5.12). An additional argument will provided in the next section.

5.3 Passage to the limit

We start with the following compactness result that is crucial in the passage to the limit in (5.10-5.12).

Proposition 5.8. *[BGPS, Proposition 1, see also VLA] For $\mathcal{O} \subset \mathbb{R}^{N+1}$, an open, connected set, let $f \in L^p((0,\tau)\times\mathcal{O}\times\mathcal{S}^N)$, $p > 1$, $\partial f/\partial t + \Omega\cdot\nabla_x f \in L^p((0,\tau)\times\mathcal{O}\times\mathcal{S}^N)$, $f|_{t=0} = f_0 \in L^\infty(\mathcal{O}\times\mathcal{S}^N)$, and $f|_{(\partial\mathcal{O}\times\mathcal{S}^N)_-} = h \in L^\infty((0,\tau)\times(\partial\mathcal{O}\times\mathcal{S}^N)_-)$. Then $\int_{\mathcal{S}^N} f(t,x,\Omega)\,d\Omega \in W^{m,p}((0,\tau)\times\mathcal{O})$ for any $0 < m < \inf(1/p, 1-1/p)$, and*

$$\left|\left|\int_{\mathcal{S}^N} f(t,x,\Omega)\,d\Omega\right|\right|_{W^{m,p}} \leq C(h,f_0)\Big[||f||_{L^p} + ||\partial f/\partial t + \Omega\cdot\nabla_x f||_{L^p}\Big]. \tag{5.19}$$

We apply the above result with $\mathcal{O} = (-a,a)$ and for $N = 0$, when $\mathcal{S}^N$ becomes the interval $[-1,1]$. Since the imbedding $W^{m,p} \subset L^p$ is compact, and $\{I(\cdot)_k\} \in L^\infty([0,\tau]\times(-a,a)\times(-1,1)) \cap L^1([0,\tau]\times(-a,a)\times(-1,1))$, uniformly in k, Proposition 5.8 implies

Theorem 5.9. *Assume that $u_0 = (T_0, n_{20}, I_0) \in D$ with $T_0 \in L^\infty(-a,a)$ and $I_0 \in L^\infty((-a,a)\times(-1,1))$. Furthermore, assume that $0 \leq I^* \in L^\infty(-1,1)$. Then, for the sequence of mild solutions $u_k(t)$, $k \geq 1$, the sequence $\{\mathcal{I}_k(\cdot)\} \in L^\infty([0,\tau]\times(-a,a)) \cap L^1([0,\tau]\times(-a,a))$ is relatively compact in $L^1([0,\tau]\times(-a,a))$.*

In particular, when $k \to \infty$, and after passing to a subsequence if necessary,

$$\mathcal{I}_k(t,x) \longrightarrow \mathcal{I}(t,x), \qquad \text{almost everywhere in } (t,x) \in [0,\tau]\times(-a,a). \tag{5.20}$$

Next, we combine (5.20) and nonnegativity of solutions to show L^1- convergence of the sequences $\{T_k(\cdot)\}\}$ and $\{n_2^k(\cdot)\}$. Indeed, after adding equation (5.11) to the equation (5.10) multiplied by 3/2, and integrating with respect to t we obtain

$$\frac{3}{2}T_k(t)+n_2^k(t)+\eta\int_0^t n_2^k(s)\left(4\pi\lambda+2\frac{\mathcal{I}_k(s)}{1+\frac{1}{k}\mathcal{I}_k(s)}\right)ds=\frac{3}{2}T_{0k}+n_{20}^k+\eta\int_0^t\frac{\mathcal{I}_k(s)}{1+\frac{1}{k}\mathcal{I}_k(s)}\,ds. \tag{5.21}$$

We note that all terms on both sides of identity (5.21) are nonnegative. Furthermore, since the sequence $\{\mathcal{I}_k(\cdot)\}\in L^\infty([0,\tau]\times(-a,a))\cap L^1([0,\tau]\times(-a,a))$ is relatively compact in $L^1([0,\tau]\times(-a,a))$, the Dominated Convergence Theorem applied to the sequence $\left\{\frac{\mathcal{I}_k(\cdot)}{1+\frac{1}{k}\mathcal{I}_k(\cdot)}\right\}$ implies that (after passing to a subsequence if necessary) the last term on the right hand side of (5.21) converges almost everywhere in $(t,x)\in[0,\tau]\times(-a,a)$. Therefore, using nonnegativity of the terms on the left hand side of (5.21), we obtain pointwise convergence almost everywhere of the sequences $\{T_k(\cdot)\}\}$ and $\{n_2^k(\cdot)\}$ in $(t,x)\in[0,\tau]\times(-a,a)$.

Thus, we have

Theorem 5.10. *Let $u_{0k}\to u_0$ in X, as $k\to\infty$, with $u_{0k}=(T_{0k},n_{20}^k,I_{0k})\in D\cap\mathcal{D}(A)$, and $T_{0k}\in L^\infty(-a,a)$, $I_{0k}\in L^\infty((-a,a)\times(-1,1))$ (uniformly in $k\geq 1$). Furthermore, assume that $0\leq I^*\in L^\infty(-1,1)$. Then, for the sequence of solutions $u_k(t)=(T_k(t),n_2^k(t),I_k(t))$ of (5.14), the sequences $\{T_k(\cdot)\}$, $\{n_2^k(\cdot)\}\in L^\infty([0,\tau]\times(-a,a))\cap L^1([0,\tau]\times(-a,a))$ are relatively compact in $L^1([0,\tau]\times(-a,a))$.*

Now, the passage to the limit can be easily achieved. The following two theorems hold:

Theorem 5.11. *Under the assumptions of Theorem 5.10, and after passing to a subsequence in $\{u_k\}$, if necessary,*

$$B_k(\cdot,u_k)\longrightarrow B(\cdot,u),\quad \textit{weakly in}\quad L^1([0,\tau]\times(-a,a)\times(-1,1)))\quad \textit{as}\quad k\to\infty, \tag{5.22}$$

where $u_k(\cdot)$ is the sequence of solutions to (5.14) converging weakly in $L^1([0,\tau]\times(-a,a)\times(-1,1)))$ to $u(\cdot)\in D$.
Furthermore, $u(\cdot):[0,\tau]\to D\subset X$ is continuous and satisfies (5.17).

Theorem 5.12. *Assume that $u_0=(T_0,n_{20},I_0)\in D$ with $T_0\in L^\infty(-a,a)$ and $I_0\in L^\infty((-a,a)\times(-1,1))$. Furthermore, let $0\leq I^*\in L^\infty(-1,1)$. Then (5.17) has a unique solution in X.*

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APPENDIX

Inelastic collision terms

$$\begin{aligned} J_1^i = & \int_{\mathbb{R}^3}\int_{\mathcal{S}} g\sigma_{12}^{11}(g,\chi)[f_1(\mathbf{v}'_+)f_1(\mathbf{w}'_+) - f_1(\mathbf{v})f_2(\mathbf{w})]d\mathbf{w}d\mathbf{\Theta}' + \\ & + 2\int_{\mathbb{R}^3}\int_{\mathcal{S}} U(g-\epsilon)g\sigma_{11}^{12}(g,\chi)[f_1(\mathbf{v}'_-)f_2(\mathbf{w}'_-) - f_1(\mathbf{v})f_1(\mathbf{w})]d\mathbf{w}d\mathbf{\Theta}' + \\ & + \int_{\mathbb{R}^3}\int_{\mathcal{S}} U(g-\epsilon)g\sigma_{12}^{22}(g,\chi)[f_2(\mathbf{v}'_-)f_2(\mathbf{w}'_-) - f_1(\mathbf{v})f_2(\mathbf{w})]d\mathbf{w}d\mathbf{\Theta}' \ , \end{aligned} \tag{A.1}$$

$$\begin{aligned} J_2^i = & \int_{\mathbb{R}^3}\int_{\mathcal{S}} g\sigma_{12}^{11}(g,\chi)[f_1(\mathbf{v}'_+)f_1(\mathbf{w}'_+) - f_2(\mathbf{v})f_1(\mathbf{w})]d\mathbf{w}d\mathbf{\Theta}' + \\ & + \int_{\mathbb{R}^3}\int_{\mathcal{S}} U(g-\epsilon)g\sigma_{12}^{22}(g,\chi)[f_2(\mathbf{v}'_-)f_2(\mathbf{w}'_-)) - f_2(\mathbf{v})f_1(\mathbf{w})]d\mathbf{w}d\mathbf{\Theta}' + \\ & + 2\int_{\mathbb{R}^3}\int_{\mathcal{S}} g\sigma_{22}^{12}(g,\chi)[f_1(\mathbf{v}'_+)f_2(\mathbf{w}'_+) - f_2(\mathbf{v})f_2(\mathbf{w})]d\mathbf{w}d\mathbf{\Theta}' \end{aligned} \tag{A.2}$$

where

$$\mathbf{v}'_\pm = \frac{1}{2}(\mathbf{v}+\mathbf{w}+g_\pm\mathbf{\Theta}'), \quad \mathbf{w}'_\pm = \frac{1}{2}(\mathbf{v}+\mathbf{w}-g_\pm\mathbf{\Theta}'),$$

$$\epsilon^2 = 4\Delta E/m, \quad g = |\mathbf{v}-\mathbf{w}|, \quad \mathbf{\Theta} = (\mathbf{v}-\mathbf{w})/g, \quad g_\pm = \sqrt{g^2 \pm \epsilon^2}, \quad \cos\chi = \mathbf{\Theta}\cdot\mathbf{\Theta}',$$

U being the Heaviside function. Note that $\mathbf{v}$ and $\mathbf{w}$ are the velocities of the colliding particles before interaction, whereas the prime indicates quantities after collision.
Moreover the following microreversibility relations link the inelastic cross–sections σ_{11}^{12}, σ_{12}^{11} and σ_{12}^{22}, σ_{22}^{12} (subscripts: incoming particles; superscripts: outcoming particles)

$$\begin{aligned} g^2\sigma_{11}^{12}(g,\chi) &= U(g-\epsilon)g_-^2\sigma_{12}^{11}(g_-,\chi) \\ g^2\sigma_{12}^{11}(g,\chi) &= g_+^2\sigma_{11}^{12}(g_+,\chi) \ , \end{aligned}$$

$$\begin{aligned} g^2\sigma_{12}^{22}(g,\chi) &= U(g-\epsilon)g_-^2\sigma_{22}^{12}(g_-,\chi) \\ g^2\sigma_{22}^{12}(g,\chi) &= g_+^2\sigma_{12}^{22}(g_+,\chi) \ . \end{aligned}$$

Radiation interaction terms

$$J_1^r = -J_2^r = \int_{\mathcal{S}} [f_2(\mathbf{v})(\alpha + \beta I(\mathbf{\Omega})) - f_1(\mathbf{v})\beta I(\mathbf{\Omega})]d\mathbf{\Omega} \ . \tag{A.3}$$

Elastic collision terms

$$J_k^e = \sum_{\ell=1}^{2}\int_{\mathbb{R}^3}\int_{\mathcal{S}} g\sigma_{k\ell}^e(g,\chi)[f_k(\mathbf{v}')f_\ell(\mathbf{w}') - f_k(\mathbf{v})f_\ell(\mathbf{w})]d\mathbf{w}d\mathbf{\Theta}' \tag{A.4}$$

$\sigma_{k\ell}^e$ being the cross–section of the elastic $k-\ell$ interaction ($\sigma_{12}^e = \sigma_{21}^e$).
Note that in (A.1–A.4), for the sake of simplicity, t and $\mathbf{x}$ have been dropped from the arguments of f_k an I.