Solutions to Homework 2

Math. 280, Spring 2008

Problem 1.
A cup of coffee in a room at temperature of 70°F is being cooled according to Newton's law of cooling:

$$\frac{dT}{dt} = -k(T-70), \qquad T(0) = 200. \quad \text{(time is measured in minutes)} \tag{1}$$

The solution to (1) is

$$T(t) = 70 + (200-70)\exp(-kt). \tag{2}$$

The coefficient k can be computed from (2) using the fact that $T(3) = 175$. Thus,

$$175 = 70 + 130\exp(-3k) \quad \Longrightarrow \quad \exp(-k) = \left(\frac{175-70}{130}\right)^{1/3} = \left(\frac{21}{26}\right)^{1/3} \tag{3}$$

The temperature of the coffee will reach 112°F at $t = t_1$ that satisfies the equation

$$112 = T(t_1) = 70 + 130\exp(-kt_1) = 70 + 130\left[\exp(-k)\right]^{t_1}, \quad \text{with } \exp(-k) \text{ given in (3)}$$

Thus,

$$\frac{112-70}{130} = \left[\left(\frac{21}{26}\right)^{1/3}\right]^{t_1}, \quad \text{or} \quad t_1 = \frac{3\log\left(\frac{21}{65}\right)}{\log\left(\frac{21}{26}\right)} \approx 15.87\,\text{minutes.}$$

Problem 2.
Let $N(t)$ denote the amount of C-14 at time t, with time $t = 0$ corresponding to the moment when the original amount N_0 of C-14 was present. Time evolution of $N(t)$ is governed by the equation

$$\frac{dN}{dt} = -\lambda N, \quad N(0) = N_0. \tag{4}$$

The amount of C-14 decays in time, thus $\lambda > 0$. From (4), $N(t) = N_0\exp(\lambda t)$ and, since the half-life of C-14 is approximately 5568 years, $N_0/2 = N_0\exp(5568\lambda)$. This gives us $\lambda = -\ln 2/5568 \approx -0.00012448$. Therefore, the expression for $N(t)$ for all time is given by

$$N(t) = N_0\exp\left(-\frac{\ln 2}{5568}t\right) = N_0 2^{-\left(\frac{t}{5568}\right)}. \tag{5}$$

At present time t, 14.5% is left, or equivalently from (5),

$$0.145N_0 = N_0 2^{-\left(\frac{t}{5568}\right)} \quad \Longrightarrow \quad \ln 0.145 = -\frac{t}{5568}\ln 2 \quad \Longrightarrow \quad t = -\frac{5568\ln 0.145}{\ln 2} \approx 15511.753 \text{ years.}$$

Problem 3.
For the equation

$$(1+x^2)dy + (xy + x^3 + x)dx = 0 \quad \text{or} \quad \frac{dy}{dx} + \left(\frac{x}{1+x^2}\right)y = -x, \tag{6}$$

an integrating factor is $\exp\left(\int \frac{x}{1+x^2}\,dx\right) = \sqrt{1+x^2}$ so that

$$\frac{d}{dx}\left[y\sqrt{1+x^2}\right] = -x\sqrt{1+x^2} \quad \Longrightarrow \quad y = -\frac{1}{3}(1+x^2) + c\left(1+x^2\right)^{-\frac{1}{2}}. \tag{7}$$

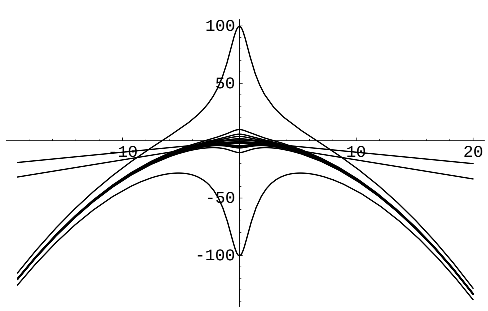

FIGURE 1. Graphs of $y = -\frac{1}{3}(1+x^2) + c\left(1+x^2\right)^{-\frac{1}{2}}$ for various choices of c.

Problem 4.

(a) For the equation

$$\left(\frac{3y^2-t^2}{y^5}\right)\frac{dy}{dt}+\frac{t}{2y^4}=0, \quad y(1)=1, \tag{8}$$

let $M=t/(2y^4)$ and $N=(3y^2-t^2)/y^5$ so that $M_y=-2t/y^5=N_t$. From $\phi_t(t,y)=M=t/(2y^4)$ we obtain $\phi(t,y)=t^2/(4y^4)+h(y)$. Thus, using $\phi_y=N=(3y^2-t^2)/y^5$, we have $h'(y)=3/y^3$, and $h(y)=-3/(2y^2)$. The general solution is $t^2/(4y^4)-3/(2y^2)=c$. If $y(1)=1$ then $c=-5/4$ and the solution to initial-value problem (8) is

$$\frac{t^2}{4y^4}-\frac{3}{2y^2}=-\frac{5}{4}. \tag{9}$$

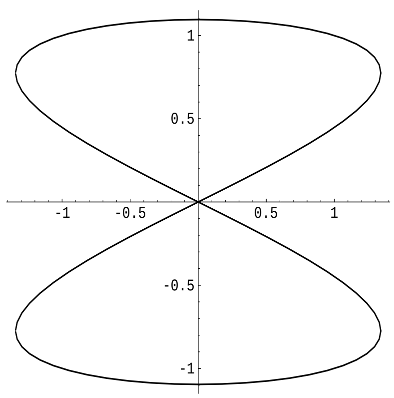

FIGURE 2. Graph of the integral curve (9)

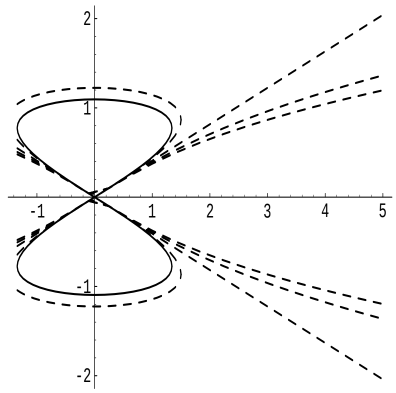

FIGURE 3. Other integral curves of $t^2/(4y^4)-3/(2y^2)=c$

The solutions of the initial value problem (8) and some other integral (dashed) curves of the differential equation

$$\left(\frac{3y^2-t^2}{y^5}\right)\frac{dy}{dt}+\frac{t}{2y^4}=0 \quad \text{are shown in Figure 3.}$$

Remark 1. *Please observe that the graph in Figure 2 does not represent a function !!! Which branch of the graph represents the solution satisfying the initial condition* $y(1)=1$.

Remark 2. *Note that if one multiplies both sides of equation* (8) *by* y^5*, the resulting equation*

$$(3y^2-t^2)\frac{dy}{dt}+\frac{1}{2}ty=0 \tag{10}$$

is not exact !!! In other words $1/y^5$ *is an integrating factor for* (10)*. (See also Problem 5, below)*

(b) The initial-value problem

$$\frac{dy}{dt}=\frac{y}{2y^2-t}, \quad y(1)=5, \quad \text{or} \quad \frac{dt}{dy}-\frac{t}{y}=2y, \quad y(1)=5, \tag{11}$$

is linear in t (as a function of y). Its integrating factor is $1/y$, thus

$$\frac{d}{dy}\left[\frac{t}{y}\right]=2 \quad \Longrightarrow \quad t=2y^2+cy.$$

If $y(1)=5$ then $c=-49/5$ and $t=2y^2-(49/5)y$.

The solutions of the initial value problem (11) and some other integral (dashed) curves of the differential equation

$$\frac{dy}{dt}=\frac{y}{2y^2-t}, \quad \text{are shown in Figure 5.}$$

Remark 3. *Please observe that the graph in Figure 4 does not represent a function !!! Which branch of the graph represents the solution satisfying the initial condition* $y(1)=5$.

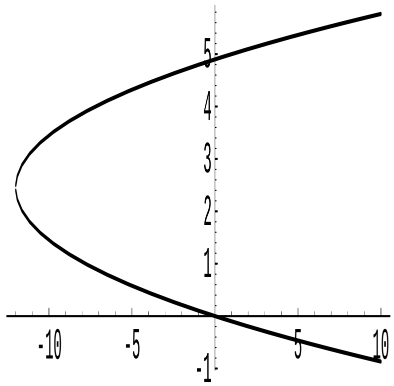

FIGURE 4. Graph of the integral curve $t = 2y^2 - (49/5)y$

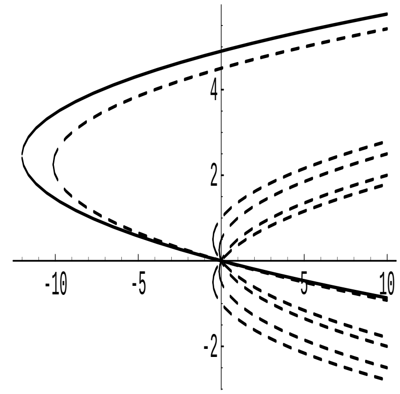

FIGURE 5. Other integral curves of $t = 2y^2 + cy$

Problem 5.
Multiply the equation

$$(t^2 + ty)\frac{dy}{dt} - y^2 = 0 \qquad \text{by} \quad \mu(t, y) = \frac{1}{t^2 y}. \quad \text{The result is} \tag{12}$$

$$\left(\frac{1}{y} + \frac{1}{t}\right)\frac{dy}{dt} - \frac{y}{t^2} = 0. \tag{13}$$

With $M = -y/t^2$ and $N = 1/y + 1/t$ we have $M_y = -1/t^2 = N_t$; therefore equation (13) is **exact**. From $\phi_t = -y/t^2$ we obtain $\phi(t, y) = y/t + h(y)$ But $\phi_y = N$; thus $h'(y) = 1/y$, and $h(y) = \log|y|$. The solution of (12) is

$$\phi(t, y) = \frac{y}{t} + \log|y| = c, \qquad \text{where } c \in \mathbb{R} \text{ is a constant.}$$

Problem 6.
Under the change of the independent variable, $t \to y$, the original equation are transformed into

$$\begin{aligned} m\frac{dv}{dt} &= -mg - kv^2, \quad k > 0 \quad \text{for the "up" motion} \\ m\frac{dv}{dt} &= mg - kv^2, \quad k > 0 \quad \text{for the "down" motion} \end{aligned} \implies \begin{aligned} mv\frac{dv}{dy} &= -mg - kv^2, \quad k > 0 \quad \text{for the "up" motion} \\ mv\frac{dv}{dy} &= mg - kv^2, \quad k > 0 \quad \text{for the "down" motion} \end{aligned} \tag{14}$$

Separating variables and integrating the differential equation corresponding to the "up" motion we have

$$\begin{aligned} \int \frac{mv}{mg + kv^2} = -\int dy \quad &\implies \quad \frac{m}{2k}\ln(mg + kv^2) = -y + c_1 \quad \implies \quad mg + kv^2 = c_2 \exp\left(-\frac{2ky}{m}\right) \\ &\implies v^2 = c_3 \exp\left(-\frac{2ky}{m}\right) - \frac{mg}{k} \end{aligned} \tag{15}$$

Using $y(0) = 0$ and $v(0) = v_0$ we have that $v = v_0$ when $y = 0$ (the initial condition) so that $v_0^2 = c_3 - mg/k$ and $c_3 = v_0^2 + mg/k$. Thus,

$$v^2 = \left(\frac{kv_0^2 + mg}{k}\right)\exp\left(-\frac{2ky}{m}\right) - \frac{mg}{k} \tag{16}$$

Setting $v = 0$ in the left hand side of (16) and solving for y we see that the maximum height is

$$h = \frac{m}{2k}\ln\left(\frac{kv_0^2 + mg}{mg}\right). \tag{17}$$

Now, separating variables and integrating the differential equation corresponding to the "down" motion (see (14)) we have

$$\begin{aligned} \int \frac{mv}{mg - kv^2} = \int dy \quad &\implies \quad -\frac{m}{2k}\ln|mg - kv^2| = y + d_1 \implies mg - kv^2 = d_2 \exp\left(-\frac{2ky}{m}\right) \\ &\implies v^2 = \frac{mg}{k}\left[1 - d_3 \exp\left(-\frac{2ky}{m}\right)\right] \end{aligned} \tag{18}$$

In this case $v = 0$ when $y = 0$ (the initial condition) so $d_3 = 1$ and

$$v^2 = \frac{mg}{k}\left[1 - \exp\left(-\frac{2ky}{m}\right)\right]. \tag{19}$$

Setting $y = h$ from (17) into (19) and solving for v we obtain that the impact velocity is

$$v_i = \frac{v_0}{\sqrt{1 + \dfrac{kv_0^2}{mg}}} < v_0. \tag{20}$$

Remark 4.

By taking $y \to \infty$ in the right hand side of (19) one obtains the limiting velocity, v_{final}, for the problem $m\,(dv/dt) = mg - kv^2$

$$v_{final}^2 = \lim_{y\to\infty} \frac{mg}{k}\left[1 - \exp\left(-\frac{2ky}{m}\right)\right] = \frac{mg}{k} \quad \Longrightarrow \quad v_{final} = \sqrt{\frac{mg}{k}} > \sqrt{\frac{mg}{k}}\sqrt{\frac{v_0^2}{v_0^2 + \frac{mg}{k}}} = v_i. \quad \text{(see, (20))}$$

See also (29), where the same result can be obtained by taking $t \to \infty$ in (29) and using $\lim_{x\to\infty} \tanh(x) = 1$.

(b) It is interesting to compare the results of Problem 6 with the same problem but without air resistance. When $k = 0$ the change variables from t to y is not necessary, however, for comparison with the case $k \neq 0$, I proceed in an exactly the same way as before. The corresponding equations are

$$\begin{array}{ll} m\dfrac{dv}{dt} = -mg, \quad \text{for the "up" motion} & \\ m\dfrac{dv}{dt} = mg, \quad \text{for the "down" motion} & \end{array} \Longrightarrow \begin{array}{ll} mv\dfrac{dv}{dy} = -mg, \quad \text{for the "up" motion} \\ mv\dfrac{dv}{dy} = mg, \quad \text{for the "down" motion} \end{array} \Longrightarrow \tag{21}$$

$$\begin{array}{l} \dfrac{v^2}{2} = -gy + c_1, \quad v|_{y=0} = v_0 \quad \text{for the "up" motion} \\ \dfrac{v^2}{2} = gy + c_2 \quad v|_{y=0} = 0 \quad \text{for the "down" motion} \end{array} \Longrightarrow \begin{array}{l} \dfrac{v^2}{2} = -gy + \dfrac{v_0^2}{2}, \quad \text{for the "up" motion} \\ \dfrac{v^2}{2} = gy \quad \text{for the "down" motion} \end{array} \tag{22}$$

Thus, the maximum height is $v_0^2/2g$. Observe that the maximum height given in (17) is less than the maximum height in the case without air resistance. Indeed,

$$\frac{m}{2k}\ln\left(\frac{kv_0^2 + mg}{mg}\right) = \frac{m}{2k}\ln\left(1 + \frac{kv_0^2}{mg}\right) < \frac{m}{2k}\left(\frac{kv_0^2}{mg}\right) \quad \left\{\begin{array}{c}\text{since for } x > 0 \\ \ln(1+x) < x\end{array}\right\} \quad = \frac{v_0^2}{2g}$$

Also the impact velocity in the case without air resistance is equal to the initial velocity v_0. indeed, substituting $v_0^2/2g$ into $v^2/g = gy$ (expression for v for the "down" part of the motion, see, (22)) we obtain

$$v_i = v_0 \qquad \textbf{(No dissipation of the energy)}$$

Additional remarks to Problem 6

Finally, I show how one can integrate the equations

$$m\frac{dv}{dt} = -mg - kv^2, \quad v(0) = v_0, \quad k > 0 \quad \text{for the "up" motion} \tag{23}$$

$$m\frac{dv}{dt} = mg - kv^2, \quad v(0) = 0, \quad k > 0 \quad \text{for the "down" motion} \tag{24}$$

without changing the variable t to y. The integrations are slightly more involved. For the "up" motion we have, after separation of variables

$$\frac{m}{k}\left(\frac{dv}{\dfrac{mg}{k} + v^2}\right) = -dt \quad \Longrightarrow \quad \frac{1}{\sqrt{\dfrac{mg}{k}}}\arctan\left(v/\sqrt{\frac{mg}{k}}\right) = -\frac{k}{m}t + c \quad \Longrightarrow \quad v(t) = \sqrt{\frac{mg}{k}}\tan\left(c_1 - \sqrt{\frac{kg}{m}}t\right) \tag{25}$$

Applying the initial conditions $v(0) = v_0$ we have

$$v_0 = \sqrt{\frac{mg}{k}}\tan c_1 \quad \Longrightarrow \quad c_1 = \arctan\left(v_0/\sqrt{\frac{mg}{k}}\right) \quad \Longrightarrow \quad v(t) = \sqrt{\frac{mg}{k}}\tan\left[\arctan\left(v_0/\sqrt{\frac{mg}{k}}\right) - \sqrt{\frac{kg}{m}}t\right]. \tag{26}$$

From (26) the time t_{up} needed for the projectile to attain its maximum height ($v(t_{up}) = 0$) is

$$t_{up} = \sqrt{\frac{m}{kg}}\arctan\left(v_0/\sqrt{\frac{mg}{k}}\right) \tag{27}$$

For the "down" motion

$$\frac{m}{k}\left(\frac{dv}{\dfrac{mg}{k} - v^2}\right) = dt \quad \Longrightarrow \quad \frac{1}{\sqrt{\dfrac{mg}{k}}}\tanh^{-1}\left(v/\sqrt{\frac{mg}{k}}\right) = \frac{k}{m}t + d_1 \quad \Longrightarrow \quad v(t) = \sqrt{\frac{mg}{k}}\tanh\left(d_1 + \sqrt{\frac{kg}{m}}t\right) \tag{28}$$

Applying the initial condition $v(0) = 0$ one obtains $d_1 = 0$ and

$$v(t) = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}} t\right) \tag{29}$$

The time t_{down} needed for the projectile to fall from the height (17) can also be computed; from (20)

$$\frac{v_0}{\sqrt{1 + \frac{kv_0^2}{mg}}} = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}} t\right) \implies \frac{v_0\sqrt{k}}{\sqrt{mg + kv_0^2}} = \tanh\left(\sqrt{\frac{kg}{m}} t\right)$$

$$\implies t_{down} = \sqrt{\frac{mk}{g}} \tanh^{-1}\left(\frac{v_0\sqrt{k}}{\sqrt{mg + kv_0^2}}\right) \tag{30}$$

One can also show that

$$t_{up} < t_{down}. \tag{31}$$

Furthermore, when air resistance is neglected

$$t_{up} = t_{down} = \frac{v_0}{g}. \qquad \textbf{(No dissipation of the energy)}$$

Problem 7.

(a) The equation for velocity at any time and position y is

$$v^2 = v_0^2 - 2gR + \frac{2gR^2}{R + y}, \tag{32}$$

where R is the radius of the earth, g is the acceleration due to gravity, and v_0 is the initial velocity of the projectile (in our case, $v_0 < \sqrt{2gR}$). Now the maximum distance, $y = Y_{max}$, is reached by the projectile when $v = 0$. In other words, we have the following equation for Y_{max}:

$$0 = V_0^2 - 2gR + \frac{2gR^2}{R + Y_{max}}. \tag{33}$$

Its solution is

$$Y_{max} = \frac{Rv_0^2}{2gR - v_0^2}.$$

(b) The solution to the equation $y'' = -g$ with the initial conditions $y(0) = 0$ and $y'(0) = v(0) = v_0 > 0$ is $y(t) = -\frac{1}{2}gt^2 + v_0 t$ (with $v(t) = -gt + v_0$). As in part (a), the maximum distance, y_{max}, is reached when $v = 0$; this corresponds to $t = v_0/g$. Thus, the maximum distance y_{max} is

$$y_{max} = y\left(\frac{v_0}{g}\right) = -\frac{1}{2}g\left(\frac{v_0}{g}\right)^2 + v_0\left(\frac{v_0}{g}\right) = \frac{v_0^2}{2g}.$$

Finally, since $v_0 < \sqrt{2gR}$, we have

$$Y_{max} = \frac{Rv_0^2}{2gR - v_0^2} = \frac{v_0^2}{2g - (v_0^2/R)} > \frac{v_0^2}{2g} = y_{max}.$$

(c) In this case (32) becomes

$$v = \frac{dy}{dt} = \sqrt{\frac{2gR^2}{R + y}}, \quad y(0) = 0, \qquad \left(\textit{Note the plus sign in front of } \sqrt{\frac{2gR^2}{R + y}}\right) \tag{34}$$

which is a separable equation. The solution of (34) is

$$\frac{2}{3}(R + y)^{3/2} = Rt\sqrt{2g} + c.$$

With $y(0) = 0$ we have $c = (2/3)\,R^{3/2}$. Thus

$$y(t) = \left(\frac{3\sqrt{2g}}{2}Rt + R^{3/2}\right)^{2/3} - R. \tag{35}$$

(d) Solving (35) for t we obtain

$$t = \frac{2}{3\sqrt{2g}R}\left([y(t) + R]^{3/2} - R^{3/2}\right). \tag{36}$$

With $R = 3963$ miles, $y(T_M) = 238,855$ miles, and $g = 32/5280 = 0.0061$ mile/s^2, we get from (36)
$T_M = 181853$ seconds $=$ 50.52 hours.