Question 1. Consider the linear system

$$\left|\begin{aligned} -3x_2+6x_3+3x_4&=-9\\ x_1-\ x_2+\ x_3-\ x_4&=1\\ 2x_1-4x_2\qquad -2x_4&=6\end{aligned}\right|$$

(a) Use the Gauss-Jordan algorithm to calculate the reduced row echelon form of its augmented matrix. You must label all row operations performed.

(b) Which are the free variables? Which are the leading variables?

(c) What is the rank of the matrix of coefficients of the system?

(d) Calculate all the solutions to the system.

Answers. (a) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & -5/3 & 7/3\\ 0 & 1 & 0 & -1/3 & -1/3\\ 0 & 0 & 1 & 1/3 & -5/3\end{array}\right]$

(b) Free vars.: x_4. Leading vars.: x_1, x_2, x_3.

(c) 3

(d) $\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix} 7/3\\ -1/3\\ -5/3\\ 0\end{bmatrix} + t\begin{bmatrix} 5/3\\ 1/3\\ -1/3\\ 1\end{bmatrix}$

□

Question 2. For the matrix

$$A=\begin{bmatrix} 4 & 2 & 6 & -8 & 0 & 4\\ 0 & 0 & 0 & 2 & 4 & 8\\ 0 & 0 & 1 & 0 & -1 & 2\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

(a) Calculate $\mathrm{rref}(A)$.

(b) Calculate a basis for the image of A.

(c) Calculate a basis for the kernel of A.

(d) What is the rank of A?

(e) What is the nullity of A?

Answers. (a) $\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 & \frac{11}{2} & 6\\ 0 & 0 & 1 & 0 & -1 & 2\\ 0 & 0 & 0 & 1 & 2 & 4\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$

(b) $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}6\\0\\1\\0\end{bmatrix}, \begin{bmatrix}4\\-1\\0\\0\end{bmatrix}.$

(c) $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 11 \\ 0 \\ -2 \\ 4 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 2 \\ 4 \\ 0 \\ -1 \end{bmatrix}.$

(d) $\text{rank}\, A = 3$.

(e) $\text{nullity}\, A = 3$.

□

Question 3. Let L be the line in $\mathbf{R}^2$ given by the equation $2x_1 - x_2 = 0$. Let $\text{proj}_L : \mathbf{R}^2 \to \mathbf{R}^2$ denote the orthogonal projection onto L, let $\text{ref}_L : \mathbf{R}^2 \to \mathbf{R}^2$ denote reflection on L, and let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) Calculate $\text{proj}_L(\vec{e}_1)$ and $\text{proj}_L(\vec{e}_2)$

(b) Calculate the matrix, A, of the linear transformation proj_L.

(c) Calculate the matrix, B, of the linear transformation ref_L.

(d) True or false: $2A - B = I_2$

Answers. (a) $\text{proj}_L(\vec{e}_1) = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix}$ and $\text{proj}_L(\vec{e}_2) = \begin{bmatrix} 2/5 \\ 4/5 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}$

(c) $B = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$

(d) True

□

Question 4. Let H be the plane in $\mathbf{R}^3$ given by the equation $x_1 - 2x_2 + 3x_3 = 0$.

(a) Calculate a basis, $\mathfrak{B}$ for H.

(b) If $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, calculate $\vec{x}$.

(c) If $\vec{x} = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix}$, calculate $[\vec{x}]_{\mathfrak{B}}$

Answers. (a) $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$.

(b) $\vec{x} = \begin{bmatrix} 7 \\ 2 \\ -1 \end{bmatrix}$

(c) $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

□

Question 5. A linear transformation T from $\mathbf{R}^2$ to $\mathbf{R}^2$ satisfies

$$T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

(a) Calculate the matrix A that represents T (that is, $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$).

(b) Let $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$ be the basis of $\mathbf{R}^2$ formed by $\vec{v}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$, and let $[\vec{x}]_{\mathfrak{B}}$ denote the $\mathfrak{B}$-coordinates of $\vec{x}$. Calculate a matrix S such that $S\,[\vec{x}]_{\mathfrak{B}} = \vec{x}$.

(c) Calculate the $\mathfrak{B}$-matrix of T, that is, the matrix B such that $T([\vec{x}]_{\mathfrak{B}}) = B\,[\vec{x}]_{\mathfrak{B}}$.

(d) If $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, what is $[\vec{x}]_{\mathfrak{B}}$?

Answers. (a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$

(b) $S = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$

(c) $B = \begin{bmatrix} -72 & -101 \\ 52 & 73 \end{bmatrix}$

(d) $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

□

Question 6. Let H be the plane in $\mathbf{R}^3$ given by the equation $x_1 - 2x_2 + 3x_3 = 0$.

(a) Calculate the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto the plane H.

(b) Calculate the reflection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ about the plane H.

Answers. (a) $\begin{bmatrix} 6/7 \\ 9/7 \\ 4/7 \end{bmatrix}$

(b) $\begin{bmatrix} 5/7 \\ 11/7 \\ 1/7 \end{bmatrix}$

□

Question 7. Let $A = \begin{bmatrix} 1 & k & k \\ 1 & 1 & k \\ 1 & 1 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(a) For which values of k is A invertible?

(b) For which values of k does the system $A\vec{x} = \vec{b}$ have exactly one solution?

(c) For which values of k does the system $A\vec{x} = \vec{b}$ have infinitely many solutions?

(d) For which values of k does the system $A\vec{x} = \vec{b}$ have no solution?

Answers. (a) $k \neq 1$.

(b) $k \neq 1$.

(c) For no k.

(d) $k = 1$.

□

Question 8. Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

(a) Calculate the characteristic equation of A

(b) Calculate the eigenvalues of A

(c) Calculate an eigenbasis for A. If an eigenbasis does not exits, explain why.

(d) Calculate an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$. If S, or D, does not exist, explain why.

Answers. (a) $\det(A - \lambda I) = 0 \implies \lambda^2 - 4\lambda - 5 = 0$.

(b) -1 and 5.

(c) Yes: $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(d) $S = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$.

□

Question 9. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

(a) Calculate the eigenvalues of A.

(b) Calculate the algebraic and the geometric multiplicity of each eigenvalue of A

(c) Calculate a basis for the eigenspace of each eigenvalue.

(d) Is A diagonalizable? Explain.

Answers. (a) 2 and 3

(b) $\text{algmu}(2) = \text{geomu}(2) = 1$; $\text{algmu}(3) = 2$, $\text{geomu}(3) = 1$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a basis for E_2; $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a basis for E_3;

(d) A is not diagonalizable because $\text{geomu}(2) + \text{geomu}(3) = 2 < n = 3$.

□

Question 10. Let A be a 4×4 matrix such that: (i) A has rank 2; (ii) two of its eigenvalues are 3 and 5.

(a) Prove that 0 is also an eigenvalue of A.

(b) Calculate the algebraic multiplicities of the eigenvalues of A.

(c) Calculate the geometric multiplicities of the eigenvalues of A.

(d) Is A diagonalizable? If "yes", find a diagonal matrix D similar to A; if "no", explain why.

Answers. (a) $\text{rank}(A) = 2 \implies \text{nullity}(A) = 4 - 2 = 2 \implies \det(A) = 0 \implies \lambda = 0$ is an eigenvalue of A of geometric multiplicity 2

(b) $\text{algemu}(0) = 2$, $\text{algemu}(3) = \text{algemu}(5) = 1$

(c) $\text{geomu}(0) = 2$, $\text{geomu}(3) = \text{geomu}(5) = 1$

(d) Yes. A is similar to $D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

□

Question 11. Let V be the span of the vectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ in $\mathbf{R}^3$.

(a) Calculate the length of $\vec{v}_1$.

(b) Are $\vec{v}_1$ and $\vec{v}_2$ orthogonal? (Explain.)

(c) Prove that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

(d) Perform the Gram-Schmidt process on $(\vec{v}_1, \vec{v}_2)$ to obtain an orthonormal basis $(\vec{u}_1, \vec{u}_2)$ for V.

(e) Calculate the matrix of the orthogonal projection onto V.

(f) Calculate the QR-factorization of the matrix $\begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$.

Answers. (a) $\|\vec{v}_1\| = 3$.

(b) Not orthogonal because $\vec{v}_1 \cdot \vec{v}_2 = 9 \neq 0$.

(c) $a\vec{v}_1 + b\vec{v}_2 = \vec{0} \implies 2a = 0$ and $a + b = 0 \implies a = b = 0$.

(d) $\vec{u}_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$.

(e) $\text{proj}_V = \begin{bmatrix} 17/18 & 2/9 & -1/18 \\ 2/9 & 1/9 & 2/9 \\ -1/18 & 2/9 & 17/18 \end{bmatrix}$

(f) $\begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/\sqrt{2} \\ 1/3 & 0 \\ 2/3 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 2\sqrt{2} \end{bmatrix}$

□

Question 12. The eigenvalues of the symmetric matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

are -1, 0, and 5.

(a) Calculate an orthonormal eigenbasis for A.

(b) Calculate an orthogonal matrix Q such that the matrix $D = Q^T AQ$ is diagonal.

(c) What are the entries $d_{11}, d_{22}, d_{33}, d_{44}$ of D?

Answers. (a) $\vec{u}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} -1/\sqrt{15} \\ -2/\sqrt{15} \\ 3/\sqrt{15} \\ -1/\sqrt{15} \end{bmatrix}$, $\vec{u}_4 = \begin{bmatrix} 1/\sqrt{10} \\ 2/\sqrt{10} \\ 2/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$.

(b) $Q = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{15} & 1/\sqrt{10} \\ 0 & -1/\sqrt{3} & -2/\sqrt{15} & 2/\sqrt{10} \\ 0 & 0 & 3/\sqrt{15} & 2/\sqrt{10} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{15} & 1/\sqrt{10} \end{bmatrix}$

(c) $d_{11} = -1, d_{22} = 0, d_{33} = 0, d_{44} = 5$

□

Question 13. The following are Multiple-Choice/Multiple-Answers questions. In a Multiple-Answers question, a correct answer and an incorrect answer cancel each other out.

(a) Which of the following statements are TRUE?

- □ For every matrix A, the kernel of A equals the kernel of $\text{rref}(A)$.
- □ For every matrix A, the image of A equals the image of $\text{rref}(A)$.
- □ For every square matrix A, $\det(A) = \det(\text{rref}(A))$.
- □ For every square matrix A, $\det(A) = 0 \iff \det(\text{rref}(A)) = 0$.

(b) Which of the following are linear transformations from $\mathbf{R}^2$ to $\mathbf{R}^2$

- ☐ $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}x_1+x_2\\x_1-x_2\end{bmatrix}$.
- ☐ $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}x_1^2+x_2^2\\x_1^2-x_2^2\end{bmatrix}$
- ☐ $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$.
- ☐ $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}$

(c) Which of the following matrices represent a rotation of $\mathbf{R}^2$ through an angle θ?

- ☐ $\begin{bmatrix}\cos(\theta) & \sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$
- ☐ $\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$
- ☐ $\begin{bmatrix}-\sin(\theta) & \cos(\theta)\\ \cos(\theta) & \sin(\theta)\end{bmatrix}$
- ☐ $\begin{bmatrix}\cos(\theta) & \sin(\theta)\\ \sin(\theta) & -\cos(\theta)\end{bmatrix}$

(d) Let $A = \begin{bmatrix}1 & -1\\ 0 & 2\\ 5 & 3\end{bmatrix}$ and $B = \begin{bmatrix}-3 & 2 & 1\\ 1 & -1 & 2\end{bmatrix}$. Calculate the product BA.

- ☐ $BA = \begin{bmatrix}-3 & 0 & 5\\ -1 & -2 & 6\end{bmatrix}$
- ☐ $BA = \begin{bmatrix}2 & 10\\ 11 & 3\end{bmatrix}$
- ☐ $BA = \begin{bmatrix}-4 & 3 & -1\\ 2 & -2 & 4\\ -12 & 7 & 11\end{bmatrix}$
- ☐ $BA = \begin{bmatrix}-3 & 1\\ 0 & -2\\ 5 & 6\end{bmatrix}$

(e) Consider the 2×2 matrix $A = \begin{bmatrix}3 & 1\\ 6 & 2\end{bmatrix}$. Which of the following vectors are eigenvectors of A?

- ☐ $\begin{bmatrix}1\\2\end{bmatrix}$.
- ☐ $\begin{bmatrix}0\\0\end{bmatrix}$.
- ☐ $\begin{bmatrix}-1\\2\end{bmatrix}$.
- ☐ $\begin{bmatrix}-1\\3\end{bmatrix}$.

(f) Which of the following are TRUE?

- ☐ A system with more variables than equations is either inconsistent or has infinitely many solutions.
- ☐ A system that has exactly one solution cannot have fewer equations than variables.
- ☐ A system with three equations and four variables is either inconsistent or has infinitely many solutions.
- ☐ If A is a 3×4 matrix of rank 3, then the linear system $A\vec{x} = \vec{b}$ has infinitely many solutions for any $\vec{b}$.

(g) Suppose that matrix B is obtained from matrix A by performing a sequence of row operations on A that include (exactly) swapping rows s times and multiplying rows by non-zero scalars $k_1, k_2, \ldots, k_r$. Which of the following are TRUE?

- ☐ $\det(B) = (-1)^s \det(A)$.
- ☐ $\det(B) = (-1)^s (k_1 \cdot \ldots \cdot k_r) \det(A)$.
- ☐ $\det(B) = \dfrac{(-1)^s}{k_1 \cdot \ldots \cdot k_r} \det(A)$.
- ☐ $\det(B) = 0 \iff \det(A) = 0$.