A review of important results for linear systems and associated with them properties of the coefficient matrices

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Important remark !!!

In this document for $n \times m$ matrix A, n denotes the number of rows and m denotes the number of columns of matrix A. And in terms of the corresponding linear system with A as the coefficient matrix, n denotes the number of equations and m denotes the number of variables.

Consistent or inconsistent

(1) *If A is $n \times m$ matrix, then a linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.*

Equivalently,

(1′) *A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is a linear combination of the column vectors of A.*

Also

(2) *If A is $n \times m$ matrix, then a linear system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$ if and only if the column vectors of A span $\mathbb{R}^n$.*

Note that in this case $m \geq n$, and additionally, $\operatorname{rank}(A) \leq \min(m, n) \leq n$.

The proof that $\operatorname{rank}(A) \leq \min(m, n)$ proceeds as follows. If A is an $n \times m$ matrix, then the dimension of the row space of A cannot exceed n, and the dimension of the column space cannot exceed m. The dimension of the row space and the dimension of the column space are both equal to the rank of A. Therefore, $\operatorname{rank}(A) \leq \min(m, n)$.

If A is $n \times m$ matrix, then $\operatorname{rank}(A)$ is equal to the number of leading $1's$ in the reduced row echelon form of A.

(3) *If A is $n \times m$ matrix, then a linear system $A\mathbf{x} = \mathbf{b}$ is inconsistent if and only if the reduced row echelon form of the augmented matrix contains the row $[0, 0, \ldots, 1]$.*

(4) *Let A be $n \times m$ matrix. If the system $A\mathbf{x} = \mathbf{b}$ is inconsistent then $\operatorname{rank}(A) < n$.*

Equivalently,

(4′) *If $\operatorname{rank}(A) = n$, then the system $A\mathbf{x} = \mathbf{b}$ is consistent.*

*Proof of **(4)**.* If the system is inconsistent, then the reduced row echelon form of the augmented matrix will contain a row $[0, 0, \ldots, 1]$, so the reduced row echelon form of A will contain a row of zeros. Since there are no leading $1's$ in that row, we find that $\operatorname{rank}(A) < n$. □

Infinitely many solutions, exactly one solution, or no solutions

If a linear system is consistent, then it has either

- *infinitely many solutions if there is at least one free variable, or*
- *exactly one solution if all variables are leading.*

(5) *If A is $n \times m$ matrix and the system $A\mathbf{x} = \mathbf{b}$ has exactly one solution, then $\operatorname{rank}(A) = m$.*

*Proof of **(5)**.* We have

$$\begin{pmatrix} \text{number of} \\ \text{free variables} \end{pmatrix} = \begin{pmatrix} \text{total number} \\ \text{of variables} \end{pmatrix} - \begin{pmatrix} \text{number of} \\ \text{leading variables} \end{pmatrix} = m - \operatorname{rank}(A).$$

Thus, if the system has exactly one solution, then there are no free variables, so that $m - \operatorname{rank}(A) = 0$ and $\operatorname{rank}(A) = m$. □

Statement **(5)** is equivalent to

(5′) *If rank$(A) < m$, then the system $A\mathbf{x} = \mathbf{b}$ has no solutions or infinitely many solutions.*

(6) *If A is $n \times m$ matrix and the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions, then* $\operatorname{rank}(A) < m$.

*Proof of **(6)**.* If the system has infinitely many solutions, then there is at least one free variable, so that $m - \operatorname{rank}(A) > 0$ and $\operatorname{rank}(A) < m$. □

Statement **(6)** is equivalent to

(6′) *If A is $n \times m$ matrix and* $\operatorname{rank}(A) = m$, *then the system $A\mathbf{x} = \mathbf{b}$ has no solutions or exactly one solution.*

(7) *If A is $n \times m$ matrix then a linear system $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b} \in \mathbb{R}^n$ if and only if the column vectors of A are linearly independent.*

Note that in this case $m \leq n$, and additionally, $\operatorname{rank}(A) \leq \min(m, n) \leq m$.

Finally, if the column vectors of A span $\mathbb{R}^n$ and the column vectors of A are also linearly independent, then the columns vectors of A form a basis in $\mathbb{R}^n$ and $n = m$. Thus, we have

(8) *If A is $n \times n$ matrix then a linear system $A\mathbf{x} = \mathbf{b}$ has unique solution for every $\mathbf{b} \in \mathbb{R}^n$ if and only if the column vectors of A form a basis of $\mathbb{R}^n$.*

Note that a matrix A in **(8)** is nonsingular, the null space of A is $\ker(A) = \{\mathbf{0}\}$ and the image of A is $\mathbb{R}^n$.

(9) If A is $n \times m$ matrix, the rank of A plus the nullity of A equals m. (Rank-Nullity Theorem)

Note that from the identity

$$\begin{pmatrix}\text{number of}\\ \text{free variables}\end{pmatrix} = \begin{pmatrix}\text{total number}\\ \text{of variables}\end{pmatrix} - \begin{pmatrix}\text{number of}\\ \text{leading variables}\end{pmatrix} = m - \operatorname{rank}(A).$$

we conclude that the number of free variables is equal to the nullity of A.

Number of equations vs. number of variables

(10) *If A is $n \times m$ matrix and a linear system has exactly one solution then there must be at least as many equations as there are variables, i.e., $m \leq n$.*

*Proof of **(10)**.* From **(5)**, $m = \operatorname{rank}(A) \leq \min(m, n) \leq n$, so that $m \leq n$ as claimed. □

Statement **(10)** is equivalent to

(10′) *A linear system with fewer equations than variables (undetermined system, i.e., when $n < m$) has either no solutions or infinitely many solutions.*

As a corollary of **(10′)**, we have

If A is $n \times m$ matrix and $n < m$, then the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

(Overdetermined systems) *These are systems that have more equations than variables, i.e., $n > m$. These systems do sometimes have solutions, but that requires one of the equations to be a linear combination of the others. If the equations are independent, then overdetermined systems are always inconsistent.*

Structure of the solution of a linear system

(11) *If A is $n \times m$ matrix, the linear system $A\mathbf{x} = \mathbf{b}$ is consistent, and $\mathbf{x}_0$ is a particular solution (i.e., $A\mathbf{x}_0 = \mathbf{b}$), then a vector $\mathbf{y}$ is a solution if and only if $\mathbf{y} = \mathbf{x}_0 + \mathbf{z}$, where $\mathbf{z} \in$ ker(A).*

Additional examples

Problem 1

Let A be an $n \times m$ matrix. Show that if A has linearly independent columns vectors, then $\ker(A) = \{\mathbf{0}\}$.

Solution

If $(x_1, x_2, \ldots, x_m)^T = \mathbf{x} \in \ker(A)$, then $A\mathbf{x} = \mathbf{0}$. Partitioning A into columns, it follows that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots x_m\mathbf{a}_m = \mathbf{0}, \qquad \text{where } A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m].$$

Since the columns of A, $\mathbf{a}_1$, $\mathbf{a}_2$, $\ldots \mathbf{a}_m$, are linearly independent, it follows that

$$x_1 = x_2 = \cdots = x_m = 0.$$

Therefore, $\mathbf{x} = \mathbf{0}$ and $\ker(A) = \{\mathbf{0}\}$.

Observe that by the Rank-Nullity Theorem, we have $\operatorname{rank}(A) = m$.

Problem 2

How many solutions will the linear system $A\mathbf{x} = \mathbf{b}$ have if $\mathbf{b}$ is in the column space and the column vectors are linearly dependent.

Solution

The system will have infinitely solutions. Indeed, by **(1)** the system $A\mathbf{x} = \mathbf{b}$ is consistent, If the column vectors are linearly dependent, then there exists scalars $c_1, c_2, \ldots, c_m$ not all zero such that

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \ldots c_m\mathbf{a}_m = \mathbf{0}.$$

It follows that $A\mathbf{c} = \mathbf{0}$, where $\mathbf{c} = (c_1, c_2, \ldots, c_m)^T$. So $\mathbf{0} \neq \mathbf{c} \in \ker(A)$. If $\mathbf{x}_0$ is any solution of $A\mathbf{x} = \mathbf{b}$, then it follows from **(11)** that for any scalar α, the vector $\mathbf{x}_0 + \alpha\mathbf{c}$ will also be a solution.

Problem 3

Let A be a $6 \times m$ matrix of rank r and let $\mathbf{b} \in \mathbb{R}^6$. For each choice of r and m that follows indicate the possibilities as to the number of solutions one could have for the linear system $A\mathbf{x} = \mathbf{b}$.

(a) $m = 7, r = 5$
Solution

A has only 5 linearly independent column vectors, so the 7 column vectors must be linearly dependent. The column space is a proper subspace of $\mathbb{R}^6$. If $\mathbf{b}$ is not in the column space, then by **(1)**, the system is inconsistent. If $\mathbf{b}$ is in the column space, then by **(1)**, the system is consistent and the reduced row echelon form will involve 2 free variables. Indeed,

$$\begin{pmatrix}\text{number of}\\ \text{free variables}\end{pmatrix} = \begin{pmatrix}\text{total number}\\ \text{of variables}\end{pmatrix} - \begin{pmatrix}\text{number of}\\ \text{leading variables}\end{pmatrix} = 7 - \operatorname{rank}(A) = 7 - 5 = 2.$$

A consistent system involving free variables will have infinitely many solutions.

(b) $m = 7, r = 6$
Solution

A has only 6 linearly independent column vectors, so column vectors will span $\mathbb{R}^6$. Since the columns vectors span $\mathbb{R}^6$, by **(2)**, the system $A\mathbf{x} = \mathbf{b}$ will be consistent for any choice of $\mathbf{b}$. In this case the reduced row echelon form will involve 1 free variable. Indeed,

$$\begin{pmatrix}\text{number of}\\ \text{free variables}\end{pmatrix} = \begin{pmatrix}\text{total number}\\ \text{of variables}\end{pmatrix} - \begin{pmatrix}\text{number of}\\ \text{leading variables}\end{pmatrix} = 7 - \operatorname{rank}(A) = 7 - 6 = 1.$$

The system will have infinitely many solutions.

(c) $m = 5, r = 5$
Solution

A has 5 linearly independent column vectors, but they will not span $\mathbb{R}^6$. If $\mathbf{b}$ is not in the column space, then by **(1)**, the system is inconsistent. If $\mathbf{b}$ is in the column space, then by **(1)**, the system is consistent and by **(6′)**, the system will have exactly one solution.

(d) $m = 5, r = 4$
Solution

The 5 column vectors are linearly dependent and they will not span $\mathbb{R}^6$. If $\mathbf{b}$ is not in the column space, then by **(1)**, the system is inconsistent. If $\mathbf{b}$ is in the column space, then by **(1)**, the system is consistent and the reduced row echelon form will involve 1 free variable. Indeed,

$$\begin{pmatrix}\text{number of}\\ \text{free variables}\end{pmatrix} = \begin{pmatrix}\text{total number}\\ \text{of variables}\end{pmatrix} - \begin{pmatrix}\text{number of}\\ \text{leading variables}\end{pmatrix} = 5 - \operatorname{rank}(A) = 5 - 4 = 1.$$

The system will have infinitely many solutions.

Problem 4

Let A be an $n \times m$ matrix with $n > m$. Let $\mathbf{b} \in \mathbb{R}^n$ and suppose $\ker(A) = \{\mathbf{0}\}$.

What can you conclude about the column vectors of A? Are they linearly independent? Do they span $\mathbb{R}^n$?
Solution

Since $\ker(A) = \{\mathbf{0}\}$,

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots x_m\mathbf{a}_m = \mathbf{0}$$

has only trivial solution $\mathbf{x} = \mathbf{0}$, and hence the column vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_m$ are linearly independent. The columns vectors cannot span $\mathbb{R}^n$ since there are only m vectors and $m < n$.

Problem 5

Let A be an $n \times m$ matrix whose rank is equal to m. If $A\mathbf{c} = A\mathbf{d}$ does this imply that $\mathbf{c} = \mathbf{d}$? What if the rank of A is less than m?
Solution

If A is an $n \times m$ matrix with rank m, then by **(9)** (the Rank-Nullity Theorem) $N(A) = \mathbf{0}\}$. So if $A\mathbf{c} = A\mathbf{d}$ then $A(\mathbf{c} - \mathbf{d}) = \mathbf{0}$ and $\mathbf{c} - \mathbf{d} \in \ker(A) = \{\mathbf{0}\}$. So $\mathbf{c} = \mathbf{d}$. If the rank of A is less than m, then by **(9)**, A will have a nontrivial kernel space, $\ker(A) \neq \{\mathbf{0}\}$. So, if $\mathbf{0} \neq \mathbf{z} \in \ker(A)$ and $\mathbf{d} = \mathbf{c} + \mathbf{z}$, then

$$A\mathbf{d} = A(\mathbf{c} + \mathbf{z}) = A\mathbf{c} + A\mathbf{z} = A\mathbf{c}$$

and $\mathbf{d} \neq \mathbf{c}$.

Problem 6

Let A be a 5×8 matrix with the rank equal to 5 and let $\mathbf{b}$ be any vector in $\mathbb{R}^5$. Explain why the system $A\mathbf{x} = \mathbf{b}$ must have infinitely many solutions?
Solution

If A is 5×8 matrix with rank equal 5, then the column space of A will be $\mathbb{R}^5$. So by **(1)**, the system $A\mathbf{x} = \mathbf{b}$ is consistent for any $\mathbf{b} \in \mathbb{R}^5$. Its reduced echelon form will involve 3 free variables. Indeed,

$$\begin{pmatrix}\text{number of}\\ \text{free variables}\end{pmatrix} = \begin{pmatrix}\text{total number}\\ \text{of variables}\end{pmatrix} - \begin{pmatrix}\text{number of}\\ \text{leading variables}\end{pmatrix} = 8 - \operatorname{rank}(A) = 8 - 5 = 3.$$

A consistent system with free variables must have infinitely many solutions (see also $(\mathbf{5'})$).

Problem 7

A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\operatorname{rank}(A|\mathbf{b}) = \operatorname{rank}(A)$.
Solution

If the system $A\mathbf{x} = \mathbf{b}$ is consistent, then by **(1)**, $\mathbf{b}$ is in the column space of A. Therefore the column space of $(A|\mathbf{b})$ will equal the column space of A. Since the rank of a matrix is equal to the dimension of the column space, it follows that the rank of $(A|\mathbf{b})$ equals the rank of A.
Conversely, $\operatorname{rank}(A|\mathbf{b}) = \operatorname{rank}(A)$, then $\mathbf{b}$ must be in the column space of A. Indeed, if $\mathbf{b}$ were not in the column space of A, then the rank of $(A|\mathbf{b})$ would be equal to $\operatorname{rank}(A) + 1$.