VALUATIONS ON TENSOR POWERS OF A DIVISION ALGEBRA

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ABSTRACT. We study the following question in this paper: If p is a prime, m a positive integer, and $S = (s_m, \ldots, s_1)$ an arbitrary sequence consisting of "Y" or "N", does there exist a division algebra of exponent p^m over a valued field (F, v) such that the underlying division algebra of the tensor power $D^{\otimes p^i}$ has a valuation extending v if and only if $s_{m-i} = Y$? We show that if such an algebra exists, then its index must be bounded below by a power of p that depends on both m and S, and we then answer the question affirmatively by constructing such an algebra of minimal index.

1. Introduction

Let (F, v) be a valued field and let D be a finite-dimensional F-central division algebra. It is known that v may or may not extend to D; moreover, the conditions under which v extends to D are well-understood (see [3] and [5]). The following question however does not seem to have been studied: Denoting by D^r the underlying division algebra of the tensor product $\bigotimes_{i=1}^{r} D$, is there any connection between whether v extends to D and whether v extends to D^r? We may restrict attention to division algebras whose exponent is p^m for some prime p

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Date: October 1, 2004.

2000 *Mathematics Subject Classification.* Primary 16A39, 16W60; Secondary 16K50, 15A18.

Key words and phrases. division algebra, Henselization, symbol algebra, valuation.

The authors wish to thank the Mathematisches Forschungsinstitut Oberwolfach for hospitality during an RIP stay, where some of this work was done. The first-named author also wishes to thank the Departments of Mathematics at University College, Dublin, Catholique Université du Louvain, and Università degli Studi di Trento, for support while part of this work was done. The second named author also wishes to thank the Faculty of Mathematics and Physics at the University of Ljubljana, Slovenia, where part of this work was done.

The second named author was supported in part by an NSF grant.

and positive integer m (see Remark 3.6 below), and we may further restrict our attention to powers D^r where r is of the form p^i (see Corollary 2.2 below). Given D of exponent p^m, we define the *valuation sequence* of D to be the sequence $(s_m, \dots, s_1)$, where $s_{m-i} = Y$ if D^{p^i} is valued, and $s_{m-i} = N$ otherwise. Our question now becomes the following: If $S = (s_m, \dots, s_1)$ is an arbitrary sequence consisting of "Y"or "N", does there exist a division algebra of exponent p^m over a valued field (F, v) such that the valuation sequence of D is S?

We first show in this paper that if such a D exists, then its index must be bounded below by a power of p that depends on both m and S, and we then answer the question affirmatively by constructing a division algebra with valuation sequence S having this minimal index. Similar constructions of algebras of index higher than the minimum are then easy generalizations of this construction (Remark 3.5).

Our field F will be a rational function field over a field k containing all p^r-th roots of unity ω_r $(r = 1, 2, \dots)$, and our division algebras will be symbol algebras $(a, b; p^n, \omega_n, F)$; this is the algebra generated by elements i and j, and subject to the relations

$$i^{p^n} = a, \quad j^{p^n} = b, \quad ji = \omega_n ij.$$

2. Preliminaries

Let k be a field, let $a_1, \dots, a_n$ be elements of k (some or all a_i possibly equal to zero), and let $x_1, \dots, x_n$ be a set of indeterminates over k. We recall the definition of the $(x_1 - a_1, \dots, x_n - a_n)$-adic valuation on $k(x_1, \dots, x_n)$: If $n = 1$, then the $(x_1 - a_1)$-adic valuation on $k(x_1)$ is the discrete valuation corresponding to the height one prime ideal $(x_1 - a_1)$ of $k[x_1]$. For $n > 1$, the $(x_1 - a_1, \dots, x_n - a_n)$-adic valuation on $k(x_1, \dots, x_n)$ is the composite of the $(x_n - a_n)$-adic valuation on $k(x_1, \dots, x_{n-1})(x_n)$ with the $(x_1 - a_1, \dots, x_{n-1} - a_{n-1})$-adic valuation on the residue field $k(x_1, \dots, x_{n-1})$. The residue field of this valuation is k while its value group is $\mathbb{Z}^n$ ordered anti-lexicographically.

We also recall that if (F, v) is a valued field and if $x_1, \dots, x_n$ are a set of indeterminates over F, then there is a natural extension $\tilde{v}$ of v to the function field $F(x_1, \dots, x_n)$, defined on polynomials $f(x_1, \dots, x_n)$ by setting $\tilde{v}(f)$ to be the minimum of the values of the coefficients of f, and extended to the whole field by $\tilde{v}(f/g) = \tilde{v}(f) - \tilde{v}(g)$. We will refer to this extension as the *standard extension of v to $F(x_1, \dots, x_n)$*.

The value group of $\tilde{v}$ is the same as that of v, while the residue field is the function field in n variables over the residue $\overline{F}$ of F under v.

Given a field F with a fixed valuation v and given an F-central division algebra D, we will say D is *valued* if v extends to D. If F_h is a Henselization of (F, v), we recall that D is valued if and only if $D_h := D \otimes_F F_h$ is a division algebra [3, Theorem 2] (and, although we will not need this, if and only if v extends *uniquely* to every field K with $F \subseteq K \subseteq D$, see [5]).

Suppose that F, v, and D are as above, and that v extends to D. We describe two extensions of (D, v) to $D \otimes_F F(x)$. Both are well-known and can be directly verified. First, if $\tilde{v}$ is the standard extension of v to $F(x)$, then $\tilde{v}$ extends to $D \otimes_F F(x)$ by the formula $\tilde{v}(\sum(d_i \otimes x^i)) = \min_i\{v(d_i)\}$. The value group of $(D \otimes_F F(x), \tilde{v})$ equals that of (D, v), while the residue is $\overline{D} \otimes_{\overline{F}} \overline{F}(x)$, where $\overline{D}$ and $\overline{F}$ are, respectively, the residues of (D, v) and (F, v). Next, if $\tilde{v}$ is the x-adic valuation on $F(x)$ composed with the valuation v on F, then $\tilde{v}$ extends to $D \otimes_F F(x)$ by the formula $\tilde{v}(\sum(d_i \otimes x^i)) = \min_i\{v(d_i), i\}$: here, the value group of $(D \otimes_F F(x), \tilde{v})$ is $\Gamma_D \times \mathbb{Z}$ ordered anti-lexicographically, where Γ_D is the value group of (D, v). The residue is simply $\overline{D}$.

We denote by $\operatorname{ind}(D)$ and $\exp(D)$ the index and exponent of a division algebra D.

Lemma 2.1. *Let (F, v) be a valued field and let D be an F-central division algebra of exponent r. If i is an integer with $\gcd(i, r) = 1$, then D is valued if and only if D^i is valued.*

Proof. We recall that if $\gcd(i, r) = 1$, then $\operatorname{ind}(D) = \operatorname{ind}(D^i)$. Furthermore, $(D_h)^i$ is the underlying division algebra of $D^i \otimes_F F_h$, and i is also relatively prime to the exponent of D_h. Thus, if D is valued, $\operatorname{ind}(D) = \operatorname{ind}(D_h)$, so

$$\operatorname{ind}(D^i) = \operatorname{ind}(D) = \operatorname{ind}(D_h) = \operatorname{ind}((D_h)^i) = \operatorname{ind}(D^i \otimes_F F_h).$$

Hence $D^i \otimes_F F_h$ is a division algebra, so D^i is valued. The converse is clear since $D = (D^i)^j$ for some j with $\gcd(j, r) = 1$; this is a consequence of the equation $\gcd(i, r) = 1$. □

We get the following immediately, which shows that while considering p-primary algebras, where p is a prime, we may restrict our question to the powers D^{p^i}:

Corollary 2.2. *Let p be a prime, and suppose that* $\exp(D) = p^m$. *If i, j are positive integers such that j is not divisible by p, then D^{jp^i} is valued if and only if D^{p^i} is valued.*

From now on, p will denote a fixed prime.

We next determine the minimal index of a p-primary division algebra having a valuation sequence $S = (s_m, \dots, s_1)$. If $s_i = N$ and $s_{i-1} = Y$ for some i, we will refer to the pair (s_i, s_{i-1}) as an NY *subpattern of* S (a similar definition applies to YN *subpatterns.*). Recall that if D is a p-primary division algebra which is not split, then $\operatorname{ind}(D) \geq p \operatorname{ind}(D^p)$ ([1, p.76, Lemma 7]).

Proposition 2.3. *Let $S = (s_m, \dots, s_1)$ be a sequence of Y's and N's, and let δ be the number of NY subpatterns of S. If D is a p-primary division algebra with valuation sequence S, then* $\operatorname{ind}(D) \geq p^{m+\delta}$.

Proof. We prove this by induction on δ. The case $\delta = 0$ is obvious. Suppose the result holds for all sequences with at most $\delta - 1$ NY subpatterns. Since the given valuation sequence contains exactly δ NY subpatterns, we may write the pattern in the form $\cdots NY \cdots = \cdots NT$, where T is a sequence whose left-most term is Y and which has exactly $\delta - 1$ NY subpatterns. Let s be the length of T. Then T is the valuation sequence of $D^{p^{m-s}}$, so by induction, $\operatorname{ind}(D^{p^{m-s}}) \geq p^{s+\delta-1}$. Let $E = D^{p^{m-s-1}}$. Then, since E has a pattern $NY\cdots$, the algebra E is not valued, while $E^p = D^{p^{m-s}}$ is valued. Thus, we see that $\operatorname{ind}(E) > \operatorname{ind}(E_h)$ while $\operatorname{ind}(E^p) = \operatorname{ind}(E_h^p)$. Since $\operatorname{ind}(E_h) \geq p \operatorname{ind}(E_h^p)$, as E_h is not split, we have

$$\operatorname{ind}(E) > \operatorname{ind}(E_h) \geq p \operatorname{ind}(E_h^p) = p \operatorname{ind}(E^p) = p \operatorname{ind}(D^{p^{m-s}}) \geq p^{s+\delta},$$

so $\operatorname{ind}(E) \geq p^{s+1+\delta}$. Finally, by repeated applications of [1, p.76, Lemma 7], we have $\operatorname{ind}(D) \geq p^{m-s-1} \operatorname{ind}(D^{p^{m-s-1}}) = p^{m-s-1} \operatorname{ind}(E)) \geq p^{m-s-1} p^{s+1+\delta} = p^{m+\delta}$, as desired. □

We refer to $p^{m+\delta}$ as the *minimal index* of an algebra with valuation sequence S.

The following easy result will be useful in our construction.

Lemma 2.4. *Let F be a field containing a primitive p^{n+1}-th root of unity ω, and let v be a valuation on F such that the characteristic of the residue field $\overline{F}$ does not equal p. Let $D = (\alpha, \beta; p^n, \omega^p, F)$, and write*

$D^{1/p}$ for the algebra $(\alpha, \beta; p^{n+1}, \omega, F)$. Suppose that D is a division algebra.

Then,

1. *The algebra $D^{1/p}$ is a division algebra.*
2. *If D is valued with respect to v, then $D^{1/p}$ is valued with respect to v.*
3. *If $\beta = c^p u$ for some $c \in F^*$ and 1-unit u, then both D and $D^{1/p}$ are not valued with respect to v.*

Proof. Since $(D^{1/p})^p = D$, we find $\operatorname{ind}(D^{1/p}) \geq p \operatorname{ind}(D)$. Since D is a division algebra, $\operatorname{ind}(D)) = p^n$ and so $\operatorname{ind}(D^{1/p}) \geq p^{n+1}$. It follows that $D^{1/p}$ is a division algebra. For (2) we work over the Henselization: If D is valued, then $D_h := D \otimes_F F_h$ is a division algebra, so (1) shows that $(D^{1/p})_h$ is a division algebra, hence $D^{1/p}$ is valued. To prove (3), we note that because of the assumption about the characteristic of $\overline{F}$, every 1-unit is a p^r-th power in F_h for any r. Hence $D_h = (\alpha, c^p; p^n, \omega^p, F_h) \sim (\alpha, c; p^{n-1}, \omega^{p^2}, F_h)$, so $\operatorname{ind}(D_h) \leq p^{n-1}$ and D_h is therefore not a division algebra. Similarly, $(D^{1/p})_h$ is not a division algebra. Thus, both D and $D^{1/p}$ are not valued. □

3. The Construction for Minimal Index

In this section we construct, given a sequence $S = (s_m, \ldots, s_1)$ with δ NY subpatterns, a valued field (F, v) and an F-central division algebra D of exponent p^m and minimal index $p^{m+\delta}$ having valuation sequence S. We introduce the following notation. Let k be a field containing primitive p^i-th roots of unity ω_i for each $i = 1, 2, \ldots$, chosen so that $\omega_i^p = \omega_{i-1}$ for all $i \geq 2$. For any $i \geq 1$, we denote by S_i the subsequence $(s_i, \ldots, s_1)$. Furthermore, we let δ_i be the number of NY subpatterns in the subsequence S_i: thus $\delta_1 = 0$ automatically, and $\delta = \delta_m$ is the total number of NY subpatterns in S. Set $n_i = i + \delta_i$, and $n = n_m = m + \delta$. Our example will then have index p^n. In addition, we let γ_i be the number of YN subpatterns in the subsequence S_i: so again, $\gamma_1 = 0$ automatically, and $\gamma = \gamma_m$ is the total number of YN subpatterns in S.

We define our division algebra inductively as follows: We let $F_0 = k(y)$, where y is an indeterminate, and let v_0 be the trivial valuation on F_0. We also let $D_0 = F_0$. We let $F_1 = F_0(x)$, where x is a new indeterminate and we let v_1 be the x-adic valuation on F_1: note that v_1

restricts to v_0 on F_0. If $s_1 = Y$ we let D_1 be the algebra $(y, x; p, w_1, F_1)$, and if $s_1 = N$, we let D_1 be the algebra $(y, 1 + x; p, w_1, F_1)$.

It is standard that in either case D_1 is a division algebra: for instance in both cases, D_1 is nicely semiramified (NSR) with respect to the y-adic valuation on F_1 (viewed as the function field in y over $k(x)$; see [2, Ex. 4.3] for NSR algebras). In the first case, D_1 is also NSR with respect to the x-adic valuation on F_1 (so v_1 extends to D_1). However, in the second case, the x-adic valuation does not extend to D_1 as $1+x$ is a 1-unit, by Lemma 2.4 above. Thus, D_1 has the valuation sequence $S_1 = (s_1)$ with respect to the valuation v_1 on F_1. Finally, note that $D_1^p = F_1 = D_0 \otimes_{F_0} F_1$.

Now assume that for some $i \geq 2$, we have inductively constructed a valued field (F_{i-1}, v_{i-1}) that is a purely transcendental extension of F_{i-2} such that v_{i-1} restricts to v_{i-2} on F_{i-2}. Assume, too, that we have constructed a symbol algebra $D_{i-1} = (y, a_{i-1}; p^{n_{i-1}}, \omega_{n_{i-1}}, F_{i-1})$ with center F_{i-1} which is a division algebra (and hence of index $p^{n_{i-1}}$), and assume that if $s_{i-1} = N$, the slot a_{i-1} is of the form a p-th power times a 1-unit with respect to v_{i-1}. Assume that $D_{i-1}^p = D_{i-2} \otimes_{F_{i-2}} F_{i-1}$, and finally, assume that D_{i-1} has the valuation sequence S_{i-1} with respect to v_{i-1}. Recalling the definition of $D_{i-1}^{1/p}$ from the statement of Lemma 2.4, we define F_i, v_i, and D_i as follows:

Case 1: $(s_i, s_{i-1}) = (Y, Y)$ or $(s_i, s_{i-1}) = (N, N)$. (Note that $\delta_i = \delta_{i-1}$, $\gamma_i = \gamma_{i-1}$, and $n_i = n_{i-1} + 1$.) We define $F_i = F_{i-1}$, $v_i = v_{i-1}$, and $D_i = D_{i-1}^{1/p}$.

Case 2: $(s_i, s_{i-1}) = (Y, N)$. (Note that $\delta_i = \delta_{i-1}$, $\gamma_i = \gamma_{i-1} + 1$, and $n_i = n_{i-1} + 1$.) We let $Z_{\gamma_i} = \{z_{\gamma_i,0}, z_{\gamma_i,1}, \dots, z_{\gamma_i,p^{n_i-1}-1}\}$ be a new set of indeterminates, and we define $F_i = F_{i-1}(Z_{\gamma_i})$. We define v_i to be the standard extension of v_{i-1} to F_i. We let u_{γ_i} be the norm from $F_i(\alpha)$ to F_i of the element $z_{\gamma_i,0} + z_{\gamma_i,1}\alpha + \cdots + z_{\gamma_i,p^{n_i-1}-1}\alpha^{p^{n_i-1}-1}$, where we have written α for $\sqrt[p^{n_i-1}]{y}$. We define D_i to be $D_{i-1}^{1/p} \otimes_{F_{i-1}} (y, u_{\gamma_i}; p^{n_i}, \omega_{n_i}, F_i)$.

Case 3: $(s_i, s_{i-1}) = (N, Y)$. (Note that $\delta_i = \delta_{i-1} + 1$, $\gamma_i = \gamma_{i-1}$, and $n_i = n_{i-1} + 2$.) We let $W_{\delta_i} = \{w_{\delta_i,0}, w_{\delta_i,1}, \dots, w_{\delta_i,p^{n_i-1}-1}\}$ be a new set of indeterminates, and we define $F_i = F_{i-1}(W_{\delta_i})$. We define v_i to be the composite of the $(w_{\delta_i,1}, \dots, w_{\delta_i,p^{n_i-1}-1})$-adic valuation on $F_{i-1}(w_{\delta_i,0})$ composed with the standard extension of v_{i-1} to $F_{i-1}(w_{\delta_i,0})$. We let t_{δ_i} be the norm from $F_i(\alpha)$ to F_i of the element $w_{\delta_i,0} + w_{\delta_i,1}\alpha +$

$\cdots + w_{\delta_i, p^{n_i-1}-1}\alpha^{p^{n_i-1}-1}$, where we have written α for $\sqrt[p^{n_i-1}]{y}$. We define D_i to be $D_{i-1}^{1/p} \otimes_{F_{i-1}} (y, t_{\delta_i}; p^{n_i}, \omega_{n_i}, F_i)$.

We are now in a position to prove the main theorem of this paper.

Theorem 3.1. *The algebra D_i defined above is isomorphic to the symbol algebra $(y, a_i; \omega_{n_i}, p^{n_i}, F_i)$ for suitable $a_i \in F_i$, where a_i is of the form a p-th power times a 1-unit with respect to v_i in the case where $s_i = N$. D_i is a division algebra (and is hence of index p^{n_i}) and satisfies $D_i^p = D_{i-1} \otimes_{F_{i-1}} F_i$. Moreover, D_i has the valuation sequence S_i with respect to the valuation v_i on F_i. In particular, D_m is a division algebra with exponent p^m and valuation sequence S with respect to the valuation v_m on F_m, and has (minimal) index $p^{m+\delta}$.*

Proof. Since D_{i-1} is of the form $(y, a_{i-1}; p^{n_{i-1}}, \omega_{n_{i-1}}, F_{i-1})$, $D_{i-1}^{1/p} \otimes_{F_{i-1}} F_i$ is the algebra $(y, a_{i-1}; p^{n_{i-1}+1}, \omega_{n_{i-1}+1}, F_i)$. In Case 1 above, since $n_{i-1} + 1 = n_i$ we find, on taking $a_i = a_{i-1}$, that D_i is indeed the symbol algebra $(y, a_i; p^{n_i}, \omega_{n_i}, F_i)$. In Case 2 as well, $n_{i-1} + 1 = n_i$, so $D_{i-1}^{1/p} \otimes_{F_{i-1}} F_i$ is the symbol algebra $(y, a_{i-1}; p^{n_i}, \omega_{n_i}, F_i)$. Coupling this with the other factor using standard symbol algebra relations, we find D_i is the symbol algebra $(y, a_{i-1}u_{\gamma_i}; p^{n_i}, \omega_{n_i}, F_i)$. We may hence take $a_i = a_{i-1}u_{\gamma_i}$ and D_i will be in the form described in the statement of the theorem. Finally, in Case 3, $n_{i-1} + 2 = n_i$, so $D_{i-1}^{1/p} \otimes_{F_{i-1}} F_i$ is the algebra $(y, a_{i-1}; p^{n_i-1}, \omega_{n_i-1}, F_i)$. But by standard symbol algebra relations, this is the algebra $(y, a_{i-1}^p; p^{n_i}, \omega_{n_i}, F_i)$. As in Case 2, coupling this with the other factor and taking $a_i = a_{i-1}^p t_{\delta_i}$, we find D_i to be in the form described in the statement.

In the case where $(s_i, s_{i-1}) = (N, N)$, it is clear from the definition of D_i, the inductive assumption about a_{i-1}, and the fact that $v_i = v_{i-1}$, that a_i is a p-th power times a 1-unit with respect to v_i. In the case $(s_i, s_{i-1}) = (N, Y)$, note that $a_i = a_{i-1}^p t_{\delta_i}$. The element t_{δ_i} can be factored as $w_{\delta_i,0}^{p^{n_i-1}}$ times something of the form 1 plus terms involving $(w_{\delta_i,1}/w_{\delta_i,0}), \ldots, (w_{\delta_i,p^{n_i-1}-1}/w_{\delta_i,0})$. Since v_i in this case has been chosen so that $w_{\delta_i,0}$ has value 0 while all of $w_{\delta_i,1}, \ldots, w_{\delta_i,p^{n_i-1}-1}$ have positive value, a_i is indeed a p-th power times a 1-unit with respect to v_i.

It is clear that $D_i^p = D_{i-1} \otimes_{F_{i-1}} F_i$ in Case 1. Note that the new factors $(y, u_{\gamma_i}; p^{n_i}, \omega_{n_i}, F_i)$ in Case 2 and $(y, t_{\delta_i}; p^{n_i}, \omega_{n_i}, F_i)$ in Case 3 both have exponent p, since their p-th powers are $(y, u_{\gamma_i}; p^{n_i-1}, \omega_{n_i-1}, F_i)$ and $(y, t_{\delta_i}; p^{n_i-1}, \omega_{n_i-1}, F_i)$ respectively, and since both u_{γ_i} and t_{δ_i} have been chosen to be norms from the field $F_i(\sqrt[p^{n_i-1}]{y})$ to F_i. Since the p-th power of these factors are split, it is clear in both these cases as well that $D_i^p = D_{i-1} \otimes_{F_{i-1}} F_i$.

We now wish to prove that D_i is a division algebra. By Lemma 2.4 above, we only need to consider Cases 2 and 3 above. Let K be the function field of the Severi-Brauer variety of $D_{i-1}^{1/p}$, which is a generic splitting field of $D_{i-1}^{1/p}$. Then K is a regular extension of F_{i-1}, and $K \cdot F_i = K(Z_{\gamma_i})$ in Case 2 and $K \cdot F_i = K(W_{\delta_i})$ in Case 3. Writing L for $K \cdot F_i$, it is sufficient to prove that $D_i \otimes_{F_i} L$ is a division algebra. Observe that $D_i \otimes_{F_i} L$ is isomorphic to $(y, u_{\gamma_i}; p^{n_i}, \omega_{n_i}, L)$ in Case 2 and to $(y, t_{\delta_i}; p^{n_i}, \omega_{n_i}, L)$ in Case 3. Note that since K is a regular extension of F_{i-1}, y will not be a p-th power in K and the extension $L(\sqrt[p^m]{y})/L$ has degree p^m for all m. The proof D_i is a division algebra in both Case 2 and Case 3 now follows from:

Proposition 3.2. *Let L be a field containing a primitive p^m-th root of unity ω_m for all m, and let $y \in L$ be such that y is not a p-th power in L. For a fixed m, let $L(U)$ be the field obtained by adjoining the new indeterminates $U = \{u_0, u_1, \ldots, u_{p^{m-1}-1}\}$ to L. Let u be the norm from $L(U)(\alpha)$ to $L(U)$ of the element $u_0 + u_1\alpha + \cdots + u_{p^{m-1}-1}\alpha^{p^{m-1}-1}$, where we have written α for $\sqrt[p^{m-1}]{y}$. Then the symbol algebra $A = (y, u; p^m, \omega_m, L(U))$ is a division algebra of index p^m and exponent p.*

Proof. It is known that there exist division algebras of index p^m and exponent p of the form $E = (y, b; p^m, \omega_m, L')$ with center some field L' that is purely transcendental over L and linearly disjoint over L from $L(U)$. (For instance, one may add a new set of indeterminates over L and consider the algebras in [4]. Note that while y was assumed in [4] to be transcendental over a subfield of L that contains sufficient roots of unity, this was not really necessary–the proofs in that paper go through as long as y is not assumed to be a p-th power, so that the extension $L(\sqrt[p^m]{y})/L$ has degree p^m for all m.) It is sufficient to prove that $A' = A \otimes_{L(U)} L'(U) = (y, u; p^m, \omega_m, L'(U))$ is a division algebra. Since E is of exponent p, $E^p \sim (y, b; p^{m-1}, \omega_{m-1}, L')$ is split, so b is a norm from $L'(\alpha)$ to L' of some element $b_0 + b_1\alpha + \cdots + b_{p^{m-1}-1}\alpha^{p^{m-1}-1}$, where α is as in the statement of the theorem, and the b_i are in L'. We now consider the $(u_0 - b_0, u_1 - b_1, \ldots, u_{p^{m-1}-1} - b_{p^{m-1}-1})$-adic valuation on $L'(U)$, with valuation ring, say, W. With respect to this valuation, y and u are units and $\overline{u} = b$, so the W-order $(y, u; p^m, \omega_m, W)$ is an Azumaya algebra with residue $(y, b; p^m, \omega_m, L')$, which is a division algebra by assumption. By [2, Ex. 2.4(i), Prop. 2.5], A' is a division algebra, and hence of index p^m. Since $A^p = (y, u; p^{m-1}, \omega_{m-1}, L(U))$ and since u is a norm from $L(\sqrt[p^{m-1}]{y})$ to $L(y)$, A^p is split, so A is of exponent p. $\square$

Continuing with the proof of Theorem 3.1, it remains to be shown that D_i has the valuation sequence S_i with respect to the valuation v_i. We first show that D_i is valued with respect to v_i iff $s_i = Y$. For, if $s_i = N$, then we have already seen that the slot a_i must be a p-th power times a 1-unit, and Lemma 2.4 then shows that D_i is not valued. For the other direction, if $s_i = Y$ and $s_{i-1} = Y$, then Lemma 2.4 shows that D_i is indeed valued. We are thus left with the $(s_i, s_{i-1}) = (Y, N)$ situation. We have the following:

Lemma 3.3. *Let $z = x$ if $s_1 = Y$ and $z = 1 + x$ if $s_1 = N$. Then, each a_i is a product of suitable p-primary powers of the polynomials z, $u_1, \ldots, u_{\gamma_i}$, and $t_1, \ldots, t_{\delta_i}$. In particular, a_i can be written as β times a 1-unit with respect to the valuation v_i, where β is a product of suitable p-primary powers of the polynomials z, $u_1, \ldots, u_{\gamma_i}$ and the monomials $w_{1,0}, \ldots, w_{\delta_i,0}$ (with the understanding that if $s_1 = N$, then β does not contain z as a factor.) In the case $(s_i, s_{i-1}) = (Y, N)$, the extension $(F_i)_h(\sqrt[p^n]{a_{i-1}})$ of the Henselization $(F_i)_h$ with respect to v_i has residue which is contained in the field $E(\sqrt[p^n]{u_1}, \ldots, \sqrt[p^n]{u_{\gamma_{i-1}}}, \sqrt[p^n]{w_{1,0}}, \ldots, \sqrt[p^n]{w_{\delta_{i-1},0}})$, where $E = k(y)(Z_1, \ldots, Z_{\gamma_i})(w_{1,0}, \ldots, w_{\delta_{i-1},0})$.*

Proof. That a_i is a product of suitable p-primary powers of the polynomials z, $u_1, \ldots, u_{\gamma_i}$, and $t_1, \ldots, t_{\delta_i}$ is clear from the recursive definition of the algebras D_i. (It may be helpful to observe that for a given $k \geq 0$, the factor u_k appears in a if there exists a $j \leq i$ for which $\gamma_j = k$. Similar considerations apply for the factors t_k.) As noted above, each t_{δ_j} can be rewritten as $w_{\delta_j,0}^{p^{n_j-1}}$ times a 1-unit, and collecting all such 1-units together, we find that a_i can indeed be factored as β times a 1-unit with β as described. (Note that if $s_1 = N$, then $z = 1 + x$, which is a 1-unit and can be coupled with the other 1-units, so β does not contain z as a factor if $s_1 = N$.)

To determine the residue of the field $(F_i)_h(\sqrt[p^n]{a_{i-1}})$ in the case $(s_i, s_{i-1}) = (Y, N)$, note that $F_i = k(y)(x)(Z_1, \ldots, Z_{\gamma_i})(W_1, \ldots, W_{\delta_{i-1}})$. The valuation v_i may be described as the $(x, w_{1,1}, \ldots, w_{\delta_{i-1},p^{n_{i-1}-1}-1})$-adic valuation on $F_i' = k(y)(x)(w_{1,0}, \ldots, w_{\delta_{i-1},0})$ extended in the standard manner to the purely transcendental extension F_i/F_i'. The factor β in a_{i-1} is a product of p-primary powers of z, $u_1, \dots, u_{\gamma_{i-1}}$, and the monomials $w_{1,0}, \dots, w_{\delta_{i-1},0}$ (with the understanding that z does not appear in β if $s_1 = N$). Also note that $(F_i)_h(\sqrt[p^n]{a_{i-1}}) = (F_i)_h(\sqrt[p^n]{\beta})$ as every 1-unit is a p-th power. The factors of β described above show that

$$(F_i)_h(\sqrt[p^n]{\beta}) \subseteq (F_i)_h(\sqrt[p^n]{x}, \sqrt[p^n]{u_1}, \dots, \sqrt[p^n]{u_{\gamma_{i-1}}}, \sqrt[p^n]{w_{\delta_1,0}}, \dots, \sqrt[p^n]{w_{\delta_{i-1},0}})$$

if $s_1 = Y$, and

$$(F_i)_h(\sqrt[p^n]{\beta}) \subseteq (F_i)_h(\sqrt[p^n]{u_1}, \dots, \sqrt[p^n]{u_{\gamma_{i-1}}}, \sqrt[p^n]{w_{\delta_1,0}}, \dots, \sqrt[p^n]{w_{\delta_{i-1},0}})$$

if $s_1 = N$. The extension $(F_i)_h(\sqrt[p^n]{x})/(F_i)_h$ is totally ramified in the $s_1 = Y$ case, while all other p^n-th root extensions are merely lifts of the corresponding p^n-th root extensions over the residue. It follows that the residue of $(F_i)_h$ is contained in the field described in the statement of the lemma. □

To show that D_i is valued, it is sufficient to show that D_i remains a division algebra over $(F_i)_h(\sqrt[p^n]{a_{i-1}})$. But over that field, D_i is just the symbol algebra $(y, u_{\gamma_i}; p^{n_i}, \omega_{n_i}, (F_i)_h(\sqrt[p^n]{a_{i-1}}))$. If W is the valuation ring of $(F_i)_h(\sqrt[p^n]{a_{i-1}})$, then D_i contains the W order $(y, u_{\gamma_i}; p^{n_i}, \omega_{n_i}, W)$, which is an Azumaya algebra with residue $(y, u_{\gamma_i}; p^{n_i}, \omega_{n_i}, \overline{W})$, where we have written $\overline{W}$ for the residue of $(F_i)_h(\sqrt[p^n]{a_{i-1}})$. If this residue algebra is a division algebra, then [2, Ex. 2.4(i), Prop. 2.5] would show that $(y, u_{\gamma_i}; p^{n_i}, \omega_{n_i}, (F_i)_h(\sqrt[p^n]{a_{i-1}}))$ is a division algebra. Since $\overline{W} \subseteq L = E(\sqrt[p^n]{u_1}, \dots, \sqrt[p^n]{u_{\gamma_{i-1}}}, \sqrt[p^n]{w_{\delta_1,0}}, \dots, \sqrt[p^n]{w_{\delta_{i-1},0}})$ by Lemma 3.3 above, it is sufficient to show that $(y, u_{\gamma_i}; p^{n_i}, \omega_{n_i}, L)$ remains a division algebra.

Recall that $E = k(y)(Z_1, \dots, Z_{\gamma_i})(w_{1,0}, \dots, w_{\delta_{i-1},0}) = E'(Z_{\gamma_i})$, where $E' = k(y)(Z_1, \dots, Z_{\gamma_{i-1}})(w_{1,0}, \dots, w_{\delta_{i-1},0})$. Note that each u_j ($j \leq i-1$) is irreducible in the polynomial ring $k[y, Z_1, \dots, Z_{\gamma_{i-1}}, w_{1,0}, \dots, w_{\delta_{i-1},0}]$; this can be seen, for example, by the fact that after adjoining $\sqrt[p^{n_j-1}]{y}$, the polynomial u_j factors into polynomials that are linear in the Z variables, and that the Galois group of the extension acts transitively on these linear factors (permuting them cyclically). It now follows from Kummer theory that y is not a p-th power in $L' = E'(\sqrt[p^n]{u_1}, \dots, \sqrt[p^n]{u_{\gamma_{i-1}}}, \sqrt[p^n]{w_{\delta_1,0}}, \dots, \sqrt[p^n]{w_{\delta_{i-1},0}})$. Since $L = L'(Z_{\gamma_i})$, Proposition 3.2 now shows that $(y, u_{\gamma_i}; p^{n_i}, \omega_{n_i}, L)$ is a division algebra, and tracing our arguments back, we find that D_i remains a division algebra over $(F_i)_h(\sqrt[p^n]{a_{i-1}})$, and hence that D_i is valued with respect to v_i.

To show that D_i has the valuation sequence S_i, now that we have shown that D_i is valued iff $s_i = Y$, we use the fact that $D_i^p = D_{i-1} \otimes_{F_{i-1}} F_i$. By induction, D_{i-1} has the valuation sequence S_{i-1} with respect to the valuation v_{i-1} on F_{i-1}. We wish to show that $D_{i-1} \otimes_{F_{i-1}} F_i$ also has the valuation sequence S_{i-1} with respect to the valuation v_i on F_i. In Case 1 above this is clear since $F_{i-1} = F_i$ and $v_{i-1} = v_i$. For the other two cases, the result follows from:

Lemma 3.4. *Let F_{i-1} be as in Cases 2 or 3 above, and let E be any division algebra with center F_{i-1}. Then E is valued with respect to v_{i-1} if and only if $\widetilde{E} = E \otimes_{F_{i-1}} F_i$ is valued with respect to v_i.*

Proof. Note that F_i is a transcendental extension of F_{i-1}, so $\widetilde{E}$ will be a division algebra. Note too that v_i restricts to v_{i-1} on F_{i-1}. If $\widetilde{E}$ is valued with respect to v_i, then the valuation on $\widetilde{E}$ restricts to a valuation on the subalgebra E, and then this valuation on E restricted to F_{i-1} must be the same as the valuation v_i on F_i restricted to F_{i-1}. By hypothesis, this is just v_{i-1}, so indeed E is valued with respect to v_{i-1}.

The other direction follows from the remarks preceding Lemma 2.1. □

The last statement of the theorem is now clear, and D_m is our desired algebra, with center the valued field (F_m, v_m). □

Remark 3.5. To get an algebra of index higher than the minimum but exhibiting the same valuation sequence, we may simply tensor the algebra D_m defined above over F_m with as many degree p symbols of the form $(\xi_i, \eta_i; p, \omega_1, F_m(\{\xi_i, \eta_i\}))$ as necessary to increase the final index—here, the ξ_i and η_i are new indeterminates. The final valuation v_m would be defined as the $(\ldots, \xi_i, \eta_i, \ldots)$-adic valuation on $F_m(\{\xi_i, \eta_i\})$ composed with the valuation v_m above on F_m. The proof that this new algebra has the valuation sequence S is easy, and follows from the fact that the algebra D_m above has this property and that the $(\ldots, \xi_i, \eta_i, \ldots)$-adic valuation on $F_m(\{\xi_i, \eta_i\})$ extends to a totally ramified valuation on the tensor product of the symbols $(\xi_i, \eta_i; p, \omega_1, F_m(\{\xi_i, \eta_i\}))$, along with an application of [3, Theorem 1].

Remark 3.6. We consider the situation for index not a prime power. Suppose that D has index $p_1^{n_1} \cdots p_r^{n_r}$, and that $D = D_1 \otimes_F \cdots \otimes_F D_r$ with $\mathrm{ind}(D_i) = p_i^{n_i}$. It is an easy consequence of [3, Cor. 4] that D is valued if and only if each D_i is valued. Furthermore, for any s, we have $D^s = D_1^s \otimes_F \cdots \otimes_F D_r^s$. We may thus restrict ourselves to algebras of prime power index.

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