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Division Algebras Whose p^i-th Powers Have Arbitrary Index

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1. Introduction.

Let p be an arbitrary prime, and let D be a division algebra of index p^n and exponent p^k $(n \geq k \geq 1)$. It is well known that for any i $(1 \leq i \leq k)$, $ind(D^{p^i}) \leq p^{n-i}$ ([A, p. 76, Lemma 7]). Let $n_k > n_{k-1} > \ldots > n_1 > n_0 = 0$ be an arbitrary sequence of integers. The question of whether there exists a division algebra D such that $ind(D) = p^{n_k}$, $ind(D^p) = p^{n_{k-1}}$, ..., $ind(D^{p^{k-1}}) = p^{n_1}$, and $ind(D^{p^k}) = p^{n_0} = 1$ (so $\exp(D) = p^k$) was first investigated by Schofield and van den Bergh ([ScvdB, Construction 2.8]), who showed by construction that such a division algebra indeed does exist. Their construction, which is a corollary to some very deep results about generic splitting fields, is iterative and involves generic division algebras at every stage, along with passage to function fields of suitable Brauer-Severi varieties. Because of the use of generic methods, their final division algebra is somewhat complicated, (for instance, its center is hard to describe explicitly), and it is reasonable to ask if elementary examples exist.

We provide in this paper an elementary construction of a division algebra with the property above. The construction is based on repeated use of symbol algebras of the form $\left(x, x^{p^n} - y\,;\, p, F_0(y)(x), \omega\right)$, where F_0 is a field containing sufficiently many roots of unity, x and y are indeterminates, n is a suitable integer, and ω is a primitive p-th root of unity. The division algebra that results from this process is actually a symbol algebra over a rational function field over F_0, and is thus somewhat amenable to computation. Further, as an offshoot, this construction yields new examples of indecomposable algebras of index p^{n_i+t} and exponent p^{i+t} ($1 \leq i \leq k$, $t = 1, 2, \ldots$).

Our basic approach is formalized in the following theorem. (If R is any ring, $Z(R)$ will denote the center of R.)

THEOREM 1. *Let F_0 be a field containing all primitive p^i-th roots of unity ($i = 1, 2, \ldots$), and let y be an indeterminate. Let $\mathcal{C}$ be a class of finite-dimensional division algebras such that*

1. *For any $D \in \mathcal{C}$, $Z(D)$ is an extension field of $F_0(y)$.*
2. *$\mathcal{C}$ contains some extension field of $F_0(y)$.*
3. *For any $D \in \mathcal{C}$, if ind$(D) = p^n$, and if*

$$E = \left(D \otimes_{Z(D)} Z(D)(x)\right) \otimes_{Z(D)(x)} \left(x, x^{p^n} - y\,;\, p, Z(D)(x), \omega\right)$$

where x is a new indeterminate, then $E \in \mathcal{C}$. (In particular, E is a division algebra of index p^{n+1}.)

4. *For any $D \in \mathcal{C}$, there exists a division algebra $\Delta \in \mathcal{C}$ such that $Z(\Delta) = Z(D)$, $\Delta^p \sim D$, and ind$(\Delta) = p \cdot$ ind(D).*

Let $n_k > n_{k-1} > \cdots > n_1 > n_0 = 0$ be a sequence of integers. Then $\mathcal{C}$ contains an algebra A such that ind$(A) = p^{n_k}$, ind$(A^p) = p^{n_{k-1}}$, ..., ind$(A^{p^{k-1}}) = p^{n_1}$, and ind$(A^{p^k}) = p^{n_0} = 1$.

PROOF: By properties (2) and (4), $\mathcal{C}$ contains a division algebra B_1 of index p. If $n_1 = 1$, we set $A_1 = B_1$. Else, let $X = \{x_1, x_2, \ldots, x_{n_1-1}\}$ be a set of new indeterminates. We set

$$A_1 = \big(B_1 \otimes_{Z(B_1)} Z(B_1)(X)\big) \otimes_{Z(B_1)(X)} \big(x_1, x_1^p - y\,;\, p, Z(B_1)(X), \omega\big) \otimes_{Z(B_1)(X)} \cdots$$
$$\otimes_{Z(B_1)(X)} \big(x_{n_1-1}, x_{n_1-1}^{p^{n_1-1}} - y\,;\, p, Z(B_1)(X), \omega\big).$$

By repeated application of property (3), $A_1 \in \mathcal{C}$, and is thus a division algebra of index p^{n_1}. It is clear that A_1 has exponent p.

Let $Z_1 = Z(A_1)$. It will be convenient to set $A_0 = Z_0 = Z_1$. Now assume that for $1 \le j \le i < k$, $A_j \in \mathcal{C}$ has been chosen such that A_j has index p^{n_j}, and if $Z(A_j) = Z_j$, then Z_j is a purely transcendental extension of Z_{j-1} and $A_j^p \sim A_{j-1} \otimes_{Z_{j-1}} Z_j$. By property (4), $\mathcal{C}$ contains an algebra B_{i+1} with center Z_i such that $B_{i+1}^p \sim A_i$ and $ind(B_{i+1}) = p^{n_i+1}$. If $n_{i+1} = n_i + 1$, we set $A_{i+1} = B_{i+1}$. Otherwise, let $s_{i+1} = n_{i+1} - n_i$, and let $U = \{u_1, u_2, \ldots, u_{s_{i+1}-1}\}$ be a set of new indeterminates. We set

$$A_{i+1} = \big(B_{i+1} \otimes_{Z_i} Z_i(U)\big) \otimes_{Z_i(U)} \big(u_1, u_1^{p^{n_1+1}} - y\,;\, p, Z_i(U), \omega\big) \otimes_{Z_i(U)} \cdots$$
$$\otimes_{Z_i(U)} \big(u_{s_{i+1}-1}, u_{s_{i+1}-1}^{p^{s_{i+1}-1}} - y\,;\, p, Z_i(U), \omega\big).$$

We have $Z_{i+1} = Z(A_{i+1}) = Z_i(U)$, and $A_{i+1}^p \sim B_{i+1}^p \otimes_{Z_i} Z_{i+1} \sim A_i \otimes_{Z_i} Z_{i+1}$. Also, property (3) shows that $A_{i+1} \in \mathcal{C}$, and is thus a division algebra of index $p^{s_{i+1}-1} \cdot ind(B_{i+1}) = p^{n_{i+1}}$.

Proceeding inductively, A_k is the desired algebra. ∎

The construction thus depends on the existence of a class of division algebras that has the properties stated in the theorem. We describe such a class in the next section.

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2. The class $\mathcal{C}$.

Our class $\mathcal{C}$ will consist of certain valued division algebras, i.e., division algebras D with a valuation $v: D \to \Gamma$, where Γ is a totally ordered abelian group. (See [JW, §1], for instance, for the basic definitions and terminology of valued division algebras.) The image of v will be denoted Γ_D. The valuation ring of D and its maximal ideal will be denoted V_D and M_D respectively, and $\overline{D}$ will denote the residue V_D/M_D.

Note that all our division algebras need to contain $F_0(y)$ (by property (1) of Theorem 1). We will restrict our attention to valued division algebras D such that

(*) the valuation on D restricts to the *trivial* valuation on $F_0(y)$.

We will find it convenient to make the following definitions. Every division algebra is *iteratively valued of height* 0. In addition, a division algebra is *iteratively valued of height* $h > 0$ if it has a valuation v on it satisfying (*) above such that $\overline{D}$ is iteratively valued of height $h-1$. (It is clear that the same division algebra can be iteratively valued of different heights h_1 and h_2 depending on the sequence of valuations chosen, but this will not be a concern.) We will denote the residue at the i-th stage by $\overline{D}^{(i)}$, and call $\overline{D}^{(h)}$ the *final residue* of D.

If $K = k(x_1, \ldots, x_n)$ is the rational function field over k in the n variables $x_1, \ldots, x_n$, and if π is any irreducible in the polynomial ring $k[x_1, \ldots, x_n]$, then the π-adic valuation on K is the discrete valuation of the ring obtained by localizing $k[x_1, \ldots, x_n]$ at the height 1 prime ideal generated by π.

Recall that if K/k is an algebraic extension of fields, then any extension to K of the trivial valuation on k is also trivial. (This follows, for instance, from [E, Corollary 13.11], since Γ_k is trivial, so Γ_K/Γ_k is torsion implies that Γ_K is torsion. But the only totally ordered torsion group is the trivial group.)

Notice that since we require our valuations to be trivial on $F_0(y)$, the residue of any division algebra in $\mathcal{C}$ will contain $F_0(y)$, and hence be of the same characteristic as F_0. Since F_0 contains primitive p^i-th roots of unity ($i = 1, 2, \ldots$), the characteristic of F_0 is not equal to p. It follows from [M, Theorem 3] that if $D \in \mathcal{C}$ is of index a power of p, then D is defectless over $Z(D)$.

We set $\omega_0 = 1$, $\omega_1 = \omega$, and for each $i = 2, 3, \ldots$, we select a primitive p^i-th root of unity ω_i such that $\omega_i^p = \omega_{i-1}$. If F is a field, then as a convention, $\big(a, b\,;\, p^0, F, \omega_0\big)$ will just be the field F.

The following lemma and its corollary will be the key to our determination of $\mathcal{C}$.

LEMMA. *Let F be an extension field of $F_0(y)$ and suppose $D = \big(y, a\,;\, p^n, F, \omega_n\big)$ (for some fixed $n \geq 0$) is an iteratively valued division algebra of height h with final residue $F_0(y^{p^{-(n)}})$. Let x be a new indeterminate. Then the algebra*

$$E = \big(D \otimes_F F(x)\big) \otimes_{F(x)} \big(x, x^{p^n} - y\,;\, p, F(x), \omega\big)$$

is an iteratively valued division algebra of height $h+1$ with final residue $F_0(y^{p^{-(n+1)}})$. Moreover,

$$E \cong \Big(y, a^p \frac{x^{p^n} - y}{x^{p^n}}\,;\, p^{n+1}, F(x), \omega_{n+1}\Big).$$

PROOF: Note that if K is any extension field of $F_0(y)$, and if v is a valuation on K that satisfies (*), then v has an extension to the rational function field $K(x)$ (that also satisfies (*)) defined as follows: $v(\sum_{i=0}^{m} k_i x^i) = \min\{v(k_i)\}$, and $v(f/g) = v(f) - v(g)$ for $f, g \in K[x]$ with $g \neq 0$. The residue $\overline{K(x)}$ is just the rational function field $\overline{K}(x)$, where by abuse of notation, we write x for its image modulo $M_{K(x)}$. (See [B, Section 10, Proposition 2].)

We will write Z_i for $Z(\overline{D}^{(i)})$ ($i = 1, \ldots, h$). It will be convenient to set

$\overline{D}^{(0)} = D$ and $Z_0 = F$. We will show by reverse induction that for $i = h, h-1, \ldots, 0$,

$$E_i = \left(\overline{D}^{(i)} \otimes_{Z_i} Z_i(x)\right) \otimes_{Z_i(x)} \left(x, x^{p^n} - y\,;\, p, Z_i(x), \omega\right)$$

is an iteratively valued division algebra of height $h - i + 1$ and final residue $F_0(y^{p^{-(n+1)}})$.

When $i = h$, $E_h = \left(x, x^{p^n} - y\,;\, p, F_0(y^{p^{-(n)}})(x), \omega\right)$. Write z for $y^{p^{-n}}$ and recall that $x^{p^n} - y$ factors into linear factors of the form $x - \omega_n^j z$ $(j = 0, 1, \ldots, p^n - 1)$ in $F_0(z)[x]$. The field $F_0(z)(x)((x^{p^n} - y)^{1/p})$ is thus totally ramified over $F_0(z)(x)$ with respect to the $(x-z)$-adic valuation on $F_0(z)(x)$. On the other hand, the field $F_0(z)(x^{1/p})$ is inertial over $F_0(z)(x)$ with residue isomorphic to $F_0(y^{p^{-(n+1)}})$. Thus, E_h is a nicely semi-ramified (NSR) division algebra with respect to the $(x-z)$-adic valuation on $F_0(z)(x)$, with residue $F_0(y^{p^{-(n+1)}})$ (see [JW, Example 4.3]). This valuation is clearly trivial on $F_0(y)$, so, in particular, E_h is iteratively valued of height 1 and final residue $F_0(y^{p^{-(n+1)}})$.

Now assume that E_i is iteratively valued of height $h-i+1$ and final residue $F_0(y^{p^{-(n+1)}})$ for some i, $h \geq i \geq 1$. Consider E_{i-1}, and let v be the valuation on $\overline{D}^{(i-1)}$ with respect to which $\overline{D}^{(i-1)}$ is iteratively valued of height $h-(i-1)$. Consider the extension of $v|_{Z_{i-1}}$ to $Z_{i-1}(x)$ described at the beginning of the proof, and denote it by w. The valuations v on $\overline{D}^{(i-1)}$ and w on $Z_{i-1}(x)$ then extend to a valuation on $\overline{D}^{(i-1)} \otimes_{Z_{i-1}} Z_{i-1}(x)$, which we also denote by v. (This follows, for instance, from [M, Theorem 1]. The residue of $\overline{D}^{(i-1)} \otimes_{Z_{i-1}} Z_{i-1}(x)$ is just $\overline{D}^{(i)} \otimes_{\overline{Z_{i-1}}} \overline{Z_{i-1}}(x) \cong \overline{D}^{(i)} \otimes_{Z_i} Z_i(x)$.) We first show that $\Delta_{i-1} = \left(x, x^{p^n} - y\,;\, p, Z_{i-1}(x), \omega\right)$ is a division algebra and that w extends to this division algebra. For, let $Z_{i-1}(x)_{hens}$ denote the henselization of $Z_{i-1}(x)$ and let S be the underlying division algebra of $(\Delta_{i-1})_{hens} = \Delta_{i-1} \otimes_{Z_{i-1}(x)} Z_{i-1}(x)_{hens}$. (Thus, $(\Delta_{i-1})_{hens} = M_n(S)$, where n is either 1 or p.) The algebra $A = \left(x, x^{p^n} - y\,;\, p, V_{Z_{i-1}(x)_{hens}}, \omega\right)$ is an Azumaya algebra over the valuation ring $V_{Z_{i-1}(x)_{hens}}$ of $Z_{i-1}(x)_{hens}$ and is an order in $(\Delta_{i-1})_{hens}$ (see [JW, Example 2.4(1)]). Thus, by [JW, Proposition 2.5], S is inertial over $Z_{i-1}(x)_{hens}$, and $A \cong M_n(V_S)$, where V_S is the valuation ring of S. Hence, $A/AM_{Z_{i-1}(x)_{hens}} \cong M_n(\overline{S})$. But $A/AM_{Z_{i-1}(x)_{hens}}$ is just $\big(x, x^{p^n} - y\,;\, p, \overline{Z_{i-1}}(x), \omega\big)$. This is a division algebra, since by the induction hypothesis, $\big(x, x^{p^n} - y\,;\, p, Z_i(x), \omega\big)$ is a division algebra, and $\overline{Z_{i-1}} \subseteq Z_i$. It follows that $n = 1$, so $(\Delta_{i-1})_{hens}$ is an (inertial) division algebra. By [M, Theorem 2], w extends to Δ_{i-1}, and $(\Delta_{i-1})_{hens}/\Delta_{i-1}$ is an immediate extension.

Now write v' for the extension of w to Δ_{i-1}. Since $\Gamma_{v'} = \Gamma_{Z_{i-1}(x)}$ (as Δ_{i-1} is inertial), we have $\Gamma_v \cap \Gamma_{v'} = \Gamma_{Z_{i-1}(x)}$. Also

$$\big(\overline{D}^{(i)} \otimes_{\overline{Z_{i-1}}} \overline{Z_{i-1}}(x)\big) \otimes_{\overline{Z_{i-1}}(x)} \big(x, x^{p^n} - y\,;\, p, \overline{Z_{i-1}}(x), \omega\big) \cong$$
$$\big(\overline{D}^{(i)} \otimes_{Z_i} Z_i(x)\big) \otimes_{Z_i(x)} \big(x, x^{p^n} - y\,;\, p, Z_i(x), \omega\big) = E_i,$$

which is a division algebra by the induction hypothesis. Since our algebras are all defectless, [M, Theorem 1] shows that E_{i-1} is a valued division algebra with a valuation that restricts to v on $\overline{D}^{(i-1)}$ and v' on Δ_{i-1}. The same theorem also shows that $\overline{E_{i-1}} = E_i$. The valuation on E_{i-1} is clearly trivial on $F_0(y)$, so E_{i-1} is iteratively valued of height $h - (i-1) + 1$ with final residue $F_0(y^{p^{-(n+1)}})$. By induction, we are done.

As for the final statement of the lemma, standard symbol algebra identities (see, for instance, [D1, Chapter 11, Lemmas 6 and 11] as well as the fact that -1 is a p^r-th power for all r show

$$\begin{aligned} E &\sim \big(y, a\,;\, p^n, F(x), \omega_n\big) \otimes_{F(x)} \big(x^{p^n}, x^{p^n} - y\,;\, p^{n+1}, F(x), \omega_{n+1}\big) \\ &\sim \big(y, a^p\,;\, p^{n+1}, F(x), \omega_{n+1}\big) \otimes_{F(x)} \Big(y, \frac{x^{p^n} - y}{x^{p^n}}\,;\, p^{n+1}, F(x), \omega_{n+1}\Big) \\ &\sim \Big(y, a^p \frac{x^{p^n} - y}{x^{p^n}}\,;\, p^{n+1}, F(x), \omega_{n+1}\Big). \end{aligned}$$

Since $ind(E) = p^{n+1}$, the similarity is actually an isomorphism. ■

Corollary. *Let F be an extension field of $F_0(y)$ and suppose $\big(y, a\,;\, p^m, F, \omega_m\big)$ is an iteratively valued division algebra of height h and final residue $F_0(y^{p^{-(m)}})$ for all $m \geq n$ (where $n \geq 0$ is a fixed integer). Let x be a new indeterminate. Then the algebra*

$$E = \Big(y, a^p \frac{x^{p^n} - y}{x^{p^n}}\,;\, p^{m+1}, F(x), \omega_{m+1}\Big)$$

is an iteratively valued division algebra of height $h+1$ and final residue $F_0(y^{p^{-(m+1)}})$ for all $m \geq n$.

Proof: We have

$$E \sim \big(\big(y, a\,;\, p^m, F, \omega_m\big) \otimes_F F(x)\big) \otimes_{F(x)} \big(x^{p^n}, x^{p^n} - y\,;\, p^{m+1}, F(x), \omega_{m+1}\big).$$

Let $u = x^{p^{n-m}}$, so $[F(u) : F(x)] = p^{m-n}$, and $u^{p^m} = x^{p^n}$. Thus,

$$\begin{aligned} &E \otimes_{F(x)} F(u) \\ &\quad\sim \big(\big(y, a\,;\, p^m, F, \omega_m\big) \otimes_F F(u)\big) \otimes_{F(u)} \big(u^{p^m}, u^{p^m} - y\,;\, p^{m+1}, F(u), \omega_{m+1}\big) \\ &\quad\sim \big(\big(y, a\,;\, p^m, F, \omega_m\big) \otimes_F F(u)\big) \otimes_{F(u)} \big(u, u^{p^m} - y\,;\, p, F(u), \omega\big). \end{aligned}$$

Since $E \otimes_{F(x)} F(u)$ has index atmost p^{m+1}, the lemma (applied to the right hand side of the similarity above) shows that $E \otimes_{F(x)} F(u)$ is an iteratively valued division algebra of height $h+1$ and final residue $F_0(y^{p^{-(m+1)}})$. Thus, E is a division algebra. Moreover, the valuations on $E \otimes_{F(x)} F(u)$ and its residues induce valuations on E and its residues that are clearly trivial on $F_0(y)$. We have $\overline{E}^{(h+1)} \subseteq F_0(y^{p^{-(m+1)}})$. On the other hand, E contains the field $F_0(y^{p^{-(m+1)}})$. Since the valuation on $F_0(y^{p^{-(m+1)}})$ is necessarily trivial, $\overline{E}^{(i)}$ will contain $F_0(y^{p^{-(m+1)}})$ for $i = 1, 2, \ldots, h+1$. Thus, $\overline{E}^{(h+1)} = F_0(y^{p^{-(m+1)}})$. ∎

We are now ready to describe our class $\mathcal{C}$.

Theorem 2. *Let F be an extension field of $F_0(y)$. The class of symbol algebras $\big(y, a\,;\, p^n, F, \omega_n\big)$ $(n \geq 0)$ such that $\big(y, a\,;\, p^m, F, \omega_m\big)$ is an iteratively valued division* *algebra with finaly residue $F_0(y^{p^{-(m)}})$ for all $m \geq n$ satisfies properties (1) through (4) of Theorem 1.*

Proof: Property (1) is trivially satisfied. As for (2), consider the algebras $E_n = \big(y, x\,;\, p^n, F_0(y)(x), \omega_n\big)$ for $n \geq 0$. For $n \geq 1$, E_n is an NSR division algebra with respect to the x-adic valuation on $F_0(y)(x)$, with residue $F_0(y^{p^{-(n)}})$ (see [JW, Example 4.3]). Since for $n = 0$, E_0 $(= F_0(y)(x)$) has residue $F_0(y)$ with respect to the x-adic valuation, we find that for all $n \geq 0$, E_n is an iteratively valued division algebra of height 1 and final residue $F_0(y^{p^{-(n)}})$. In particular, $F_0(y)(x) \in \mathcal{C}$. Property (3) follows from the last statement of the lemma and the corollary. As for (4), if $D = \big(y, a\,;\, p^n, F, \omega_n\big)$, then $\Delta = \big(y, a\,;\, p^{n+1}, F, \omega_{n+1}\big)$ is also in $\mathcal{C}$, and Δ satisfies $\Delta^p \sim D$ and $ind(\Delta) = p \cdot ind(D)$. ■

Remark: If $E_0 = \big(y, a\,;\, p^n, F, \omega_n\big) \in \mathcal{C}$ has exponent p^t $(t \leq n)$, then for all $i > 0$, $E_i = \big(y, a\,;\, p^{n+i}, F, \omega_{n+i}\big)$ is an indecomposable division algebra of exponent p^{t+i}. For, when $i > 0$, $E_i^p \sim E_{i-1}$, so $ind(E_i) = p^{n+i}$ and $ind(E_i^p) = ind(E_{i-1}) = p^{n+i-1}$. By Saltman's criterion ([S, Lemma 3.2]), each E_i $(i > 0)$ is indeed indecomposable. The assertion about the exponent of E_i is clear.

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