

Pixel & Voxel Representations of Graphs

Md. Jawaherul Alam

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Ignaz Rutter
Torsten Ueckerdt*

Alexander Wolff

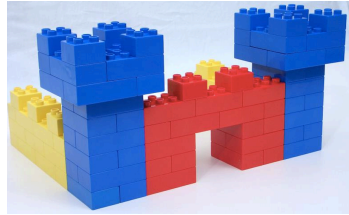


Graph Drawing
Northridge, Los Angeles – September 26, 2015

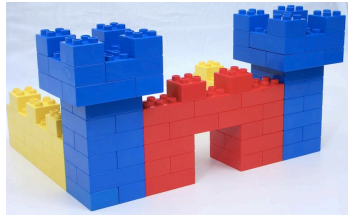
Motivation



Motivation

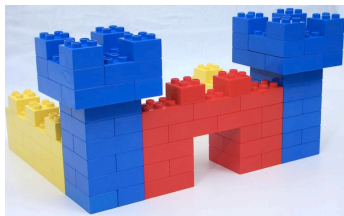
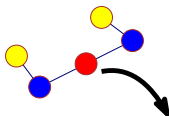


Motivation



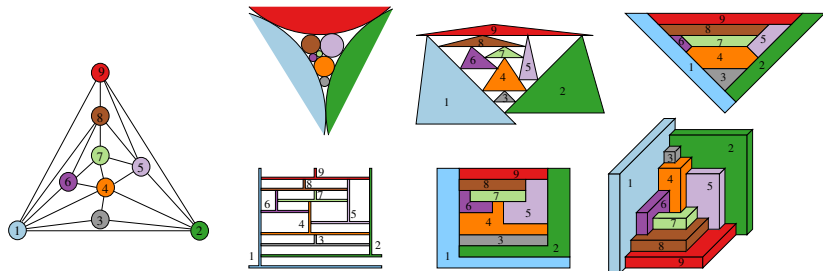
Build **contact representation** of graphs

Motivation



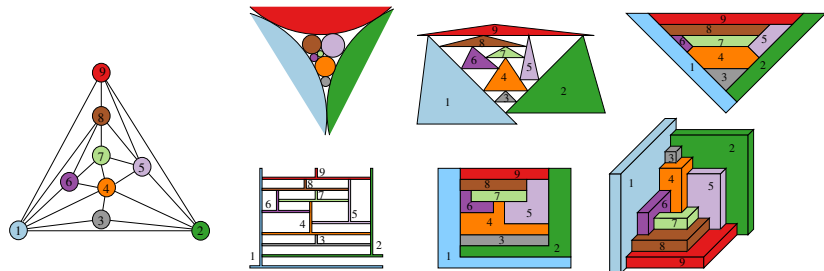
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Contact Representations



- Vertices \Rightarrow Geometric objects (polygons, arcs, polyhedra)
- Edges \Rightarrow Contacts

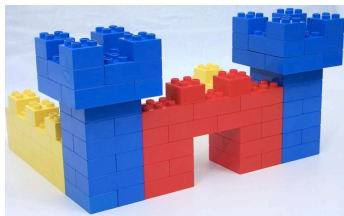
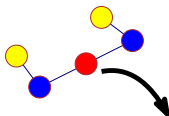
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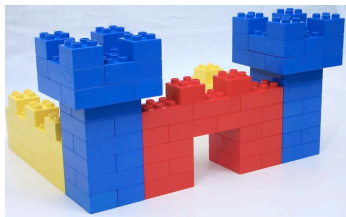
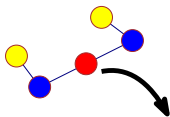
Goal: minimize polygonal complexity

Motivation



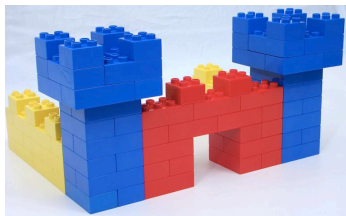
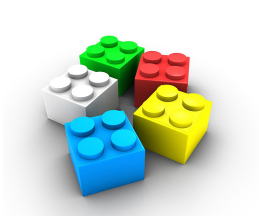
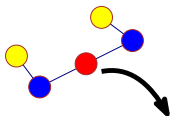
Build **contact representation** of graphs

Motivation



Build **contact representation** of graphs from **unit blocks**

Motivation



Build **contact representation** of graphs from **unit blocks**

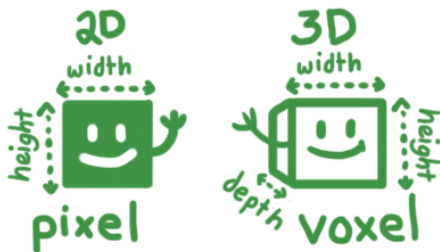
How many unit blocks are required?

Pixel and Voxel Representations

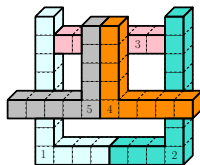
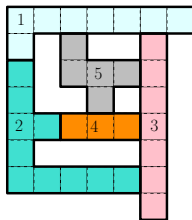
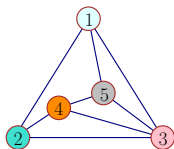
- Building contact representation from [unit blocks](#)

Pixel and Voxel Representations

- Building contact representation from [unit blocks](#)
- Pixel in 2D, Voxel in 3D

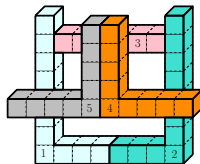
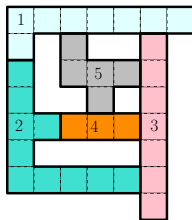
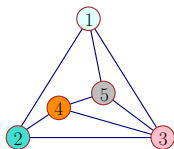


Pixel and Voxel Representations



- Vertices \Rightarrow Blobs (connected sets of pixels/voxels)
- Edges \Rightarrow Adjacent (face-to-face) pixels/voxels in two blobs

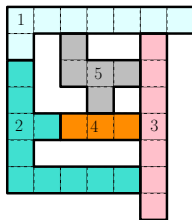
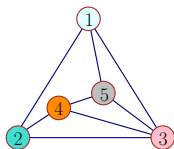
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Goal: minimize total number of pixels/voxels

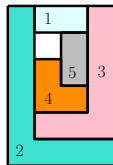
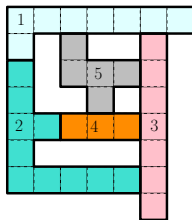
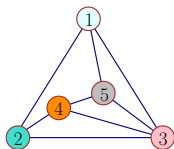
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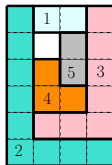
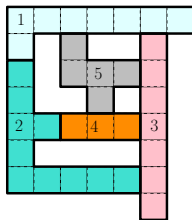
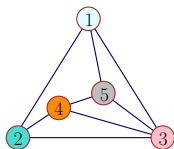
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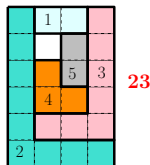
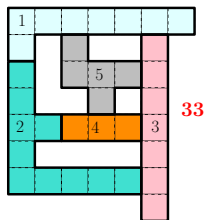
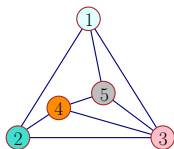
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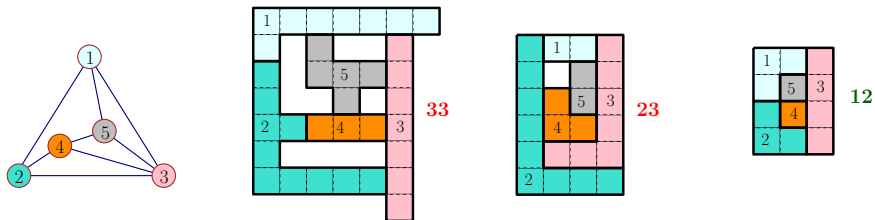
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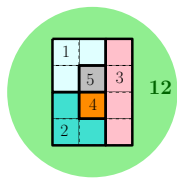
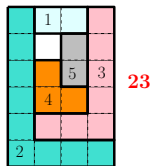
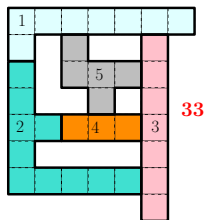
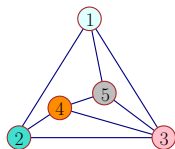
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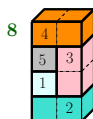
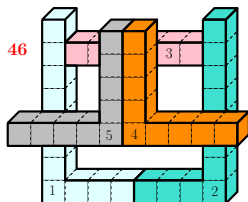
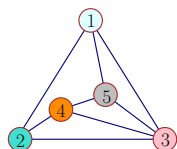
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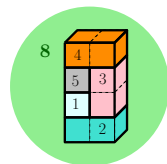
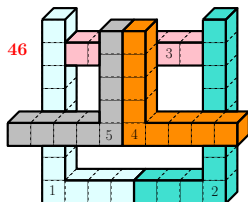
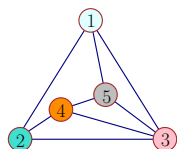
Voxel Representations



- Vertices \Rightarrow Blobs (connected sets of **voxel**)
- Edges \Rightarrow Adjacent (face-to-face) **voxels** in two blobs

Goal: minimize total number of **voxels**

Voxel Representations



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- Edges \Rightarrow Adjacent (face-to-face) **voxels** in two blobs

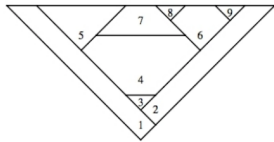
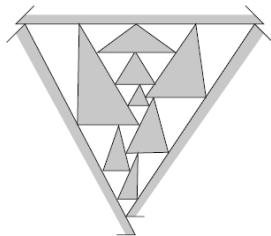
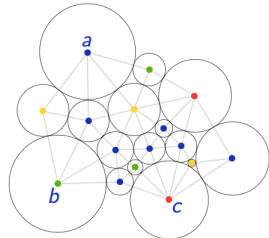
Goal: minimize total number of **voxels**

Related Work

Related Work

Contact Representations

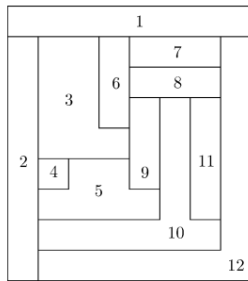
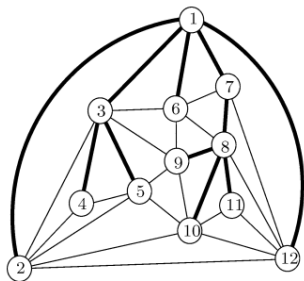
- Point-contact with circles [Koebe, 1936]
- Point-contact with triangles [De Fraysseix et al., 1994]
- Side-contact with hexagons [Gansner et al., 2010], [Bonichon et al., 2010]



Related Work

Contact Representations with Rectilinear Polygons

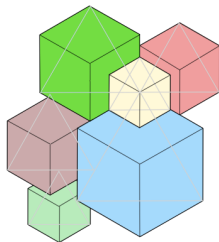
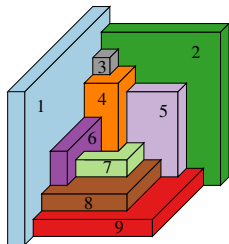
- Contact with 8-sided rectilinear polygons:
[Yeap and Sarrafzadeh, 1993], [He, 1999], [Liao et al., 2003]



Related Work

Contact Representations in 3D

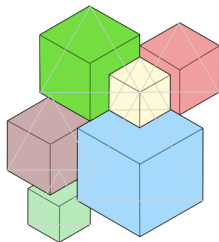
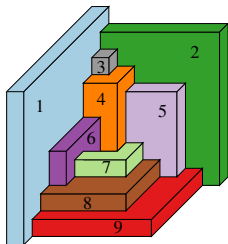
- Contact representation of planar graphs with cuboids [Thomassen, 1986], [Bremner et al., 2012]
- Improper contact representation of planar graphs with cubes [Felsner and Francis, 2011]



Related Work

Contact Representations in 3D

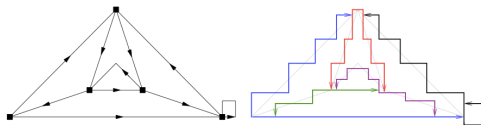
- Contact representation of planar graphs with cuboids [Thomassen, 1986], [Bremner et al., 2012]
- Improper contact representation of planar graphs with cubes [Felsner and Francis, 2011]
- Contact Representation of nonplanar graphs



Related Work

Vertex Contact Graphs of Paths on a Grid (VCPG)

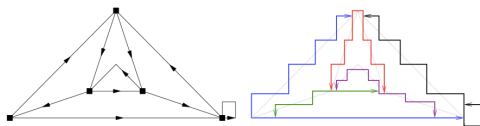
- Contact graphs of grid paths [Aerts and Felsner, 2014]



Related Work

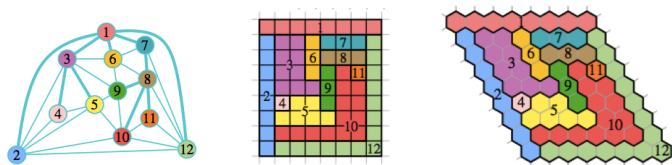
Vertex Contact Graphs of Paths on a Grid (VCPG)

- Contact graphs of grid paths [Aerts and Felsner, 2014]



Mosaic Drawing

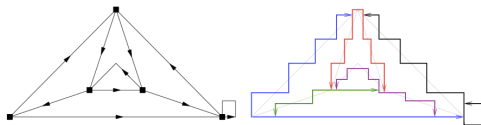
- Contact of square or hexagonal tiles [Cano et al., 2015]



Related Work

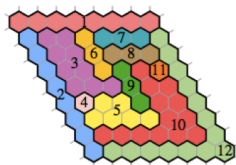
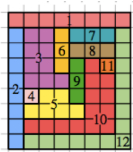
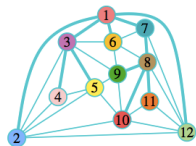
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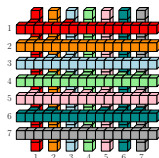
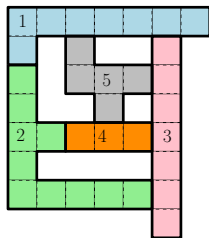


Same representation, different objective!

Our Result

Computational Complexity

- Finding minimum-size representation is NP-complete in both 2D and 3D



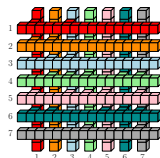
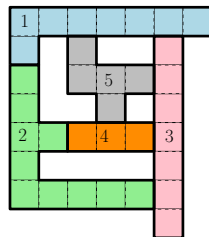
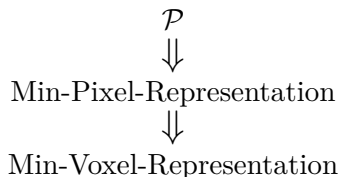
Our Result

Computational Complexity

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Reduction from: \mathcal{P}

Input: a planar max-degree-4 graph G
Find a grid drawing with unit edge lengths



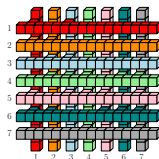
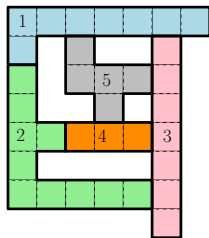
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Pixel Representation

- For a k -outerplanar graph, $\Theta(kn)$ pixels are necessary and sufficient



Our Result

Computational Complexity

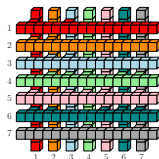
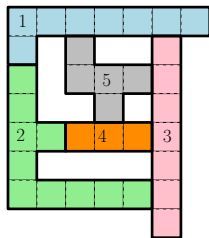
- Finding minimum-size representation is NP-complete in both 2D and 3D

Pixel Representation

- For a k -outerplanar graph, $\Theta(kn)$ pixels are necessary and sufficient

Voxel Representation

- $O(n^2)$ voxels are sufficient
- For a graph with treewidth τ , $\Theta(n \cdot \tau)$ voxels are necessary and sufficient

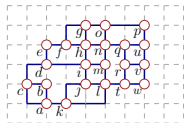


Orthogonal Drawing to Pixel/Voxel Representation

A graph G with n vertices, m edges, and an orthogonal drawing of total edge length l
 \Rightarrow Pixel/voxel representation of G with size $2l + n - m$.

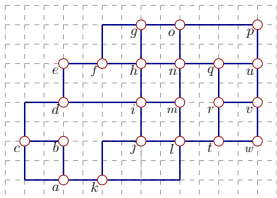
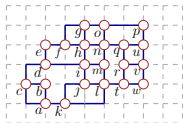
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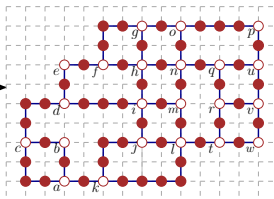
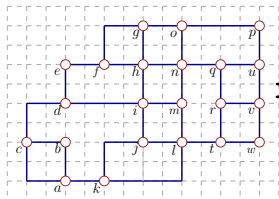
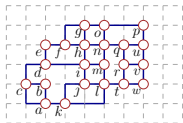
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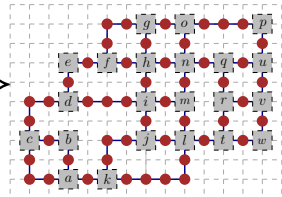
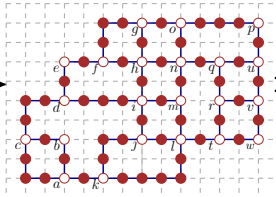
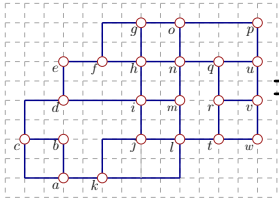
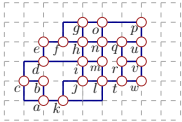
Orthogonal Drawing to Pixel/Voxel Representation

A graph G with n vertices, m edges, and an orthogonal drawing of total edge length l
 \Rightarrow Pixel/voxel representation of G with size $2l + n - m$.



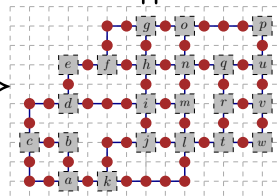
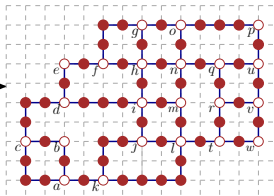
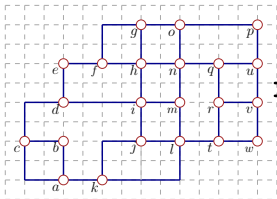
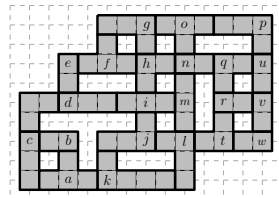
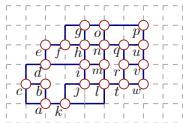
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Our Result

Computational Complexity

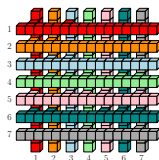
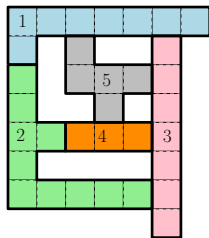
- Finding minimum-size representation is NP-complete in both 2D and 3D

Pixel Representation

- For a k -outerplanar graph, $\Theta(kn)$ pixels are necessary and sufficient

Voxel Representation

- $O(n^2)$ voxels are sufficient
- For a graph with treewidth τ , $\Theta(n \cdot \tau)$ voxels are necessary and sufficient
- For a graph with genus g , $O((g + 1)^2 n \log^2 n)$ voxels are sufficient



Our Result

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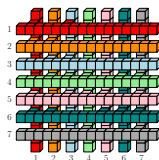
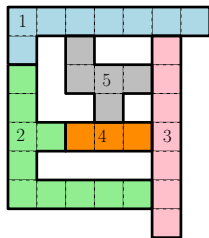
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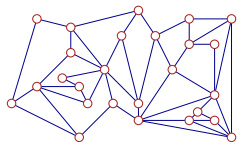
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Voxel Representation

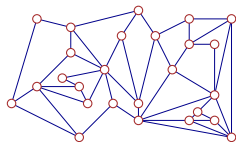
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Pixel Representations for k -Outerplanar Graphs

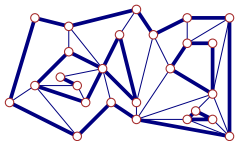


Pixel Representations for k -Outerplanar Graphs



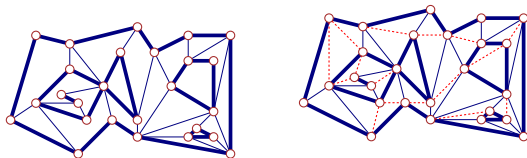
- An outerplanar graph is a 1-Outerplanar graph.
- Removing outervertices from a k -outerplanar graph yields $(k - 1)$ -outerplanar graphs

Pixel Representations for k -Outerplanar Graphs



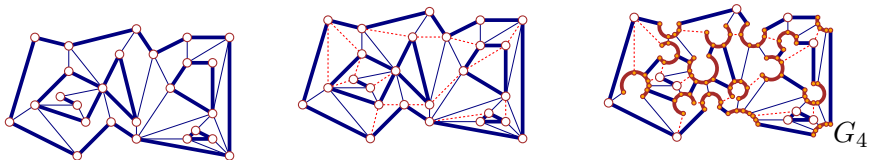
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Pixel Representations for k -Outerplanar Graphs



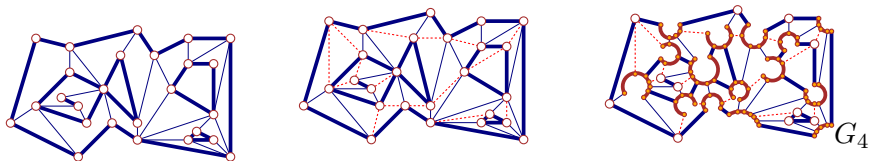
■ triangulate

Pixel Representations for k -Outerplanar Graphs



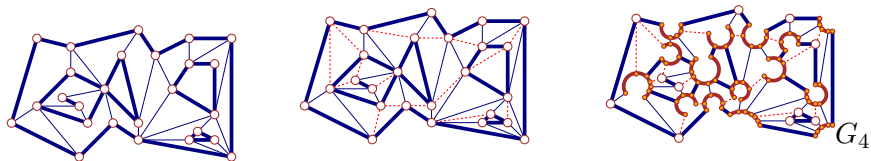
- G_4 : k -outerplanar with max-degree 4

Pixel Representations for k -Outerplanar Graphs



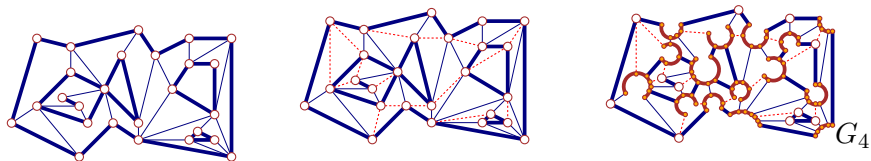
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Pixel Representations for k -Outerplanar Graphs



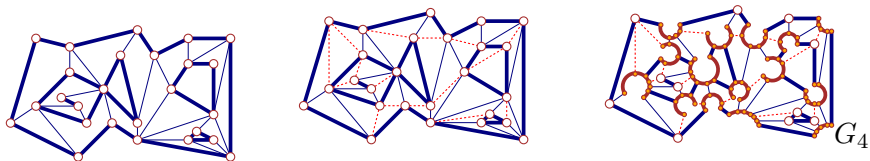
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- $\Rightarrow G_4$ has an orthogonal drawing with total edge length $\Theta(kn)$
[D. Dolev, T. Leighton, H. Trickey, 1984]

Pixel Representations for k -Outerplanar Graphs



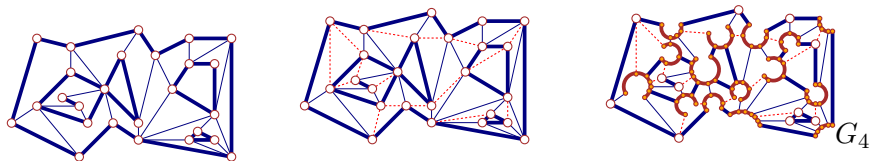
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Pixel Representations for k -Outerplanar Graphs



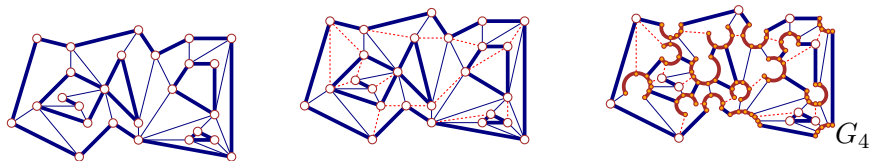
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 - Representation for G ?

Pixel Representations for k -Outerplanar Graphs



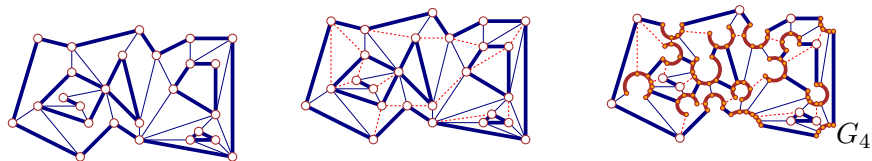
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 - Representation for G ?
 - Contract edges
 - Delete extra edges

Pixel Representations for k -Outerplanar Graphs



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Pixel Representations for k -Outerplanar Graphs



- G_4 : k -outerplanar with max-degree 4
- Maximum shortest path length to reach outerface = $\Theta(k)$
- ⇒ G_4 has an orthogonal drawing with total edge length $\Theta(kn)$
[D. Dolev, T. Leighton, H. Trickey, 1984]
- ⇒ G_4 has a pixel representation with size $\Theta(kn)$
 - **Representation for G ?**
 - Contract edges: identify blobs
 - Delete extra edges: remove contact pixels

Pixel Representations for k -Outerplanar Graphs

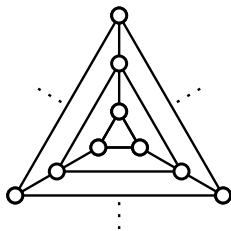
Lower Bound

- Any k -outerplane pixel representation has size at least $4k^2 - 4k$.

Pixel Representations for k -Outerplanar Graphs

Lower Bound

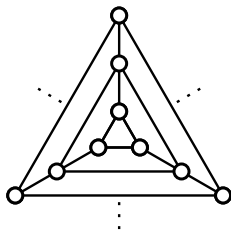
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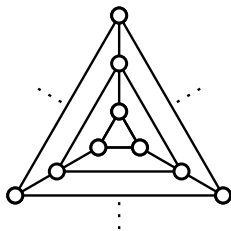


- ⇒ $\Theta(kn)$ pixels are sometimes necessary and always sufficient

Pixel Representations for k -Outerplanar Graphs

Lower Bound

- Any k -outerplane pixel representation has size at least $4k^2 - 4k$.
- ⇒ Some k -outerplanar graphs require $\Omega(kn)$ pixels



- ⇒ $\Theta(kn)$ pixels are sometimes necessary and always sufficient
- Linear pixels for outerplanar, quadratic for planar graphs.

Our Result

Computational Complexity

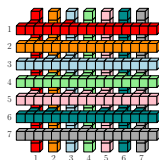
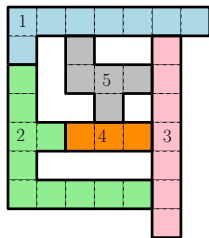
- Finding minimum-size representation is NP-complete in both 2D and 3D

Pixel Representation

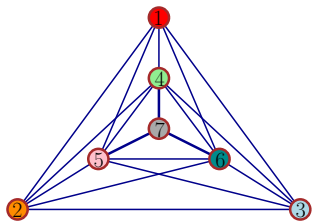
- For a k -outerplanar graph, $\Theta(kn)$ pixels are necessary and sufficient

Voxel Representation

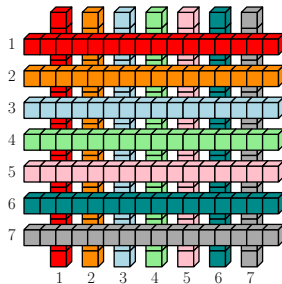
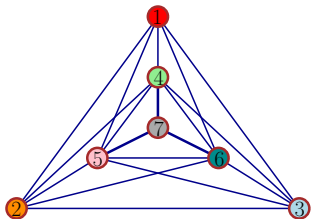
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Voxel Representations for Graphs



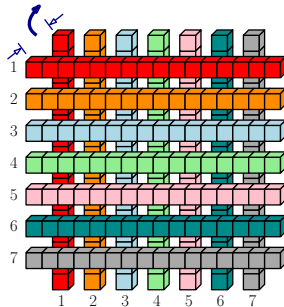
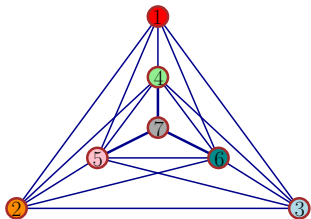
Voxel Representations for Graphs



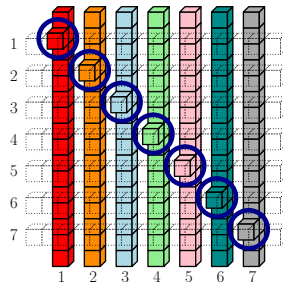
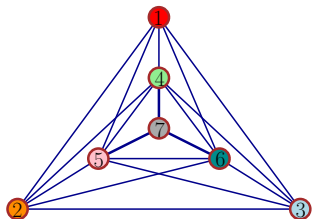
Voxel Representations for Graphs



NO TOUCHING!

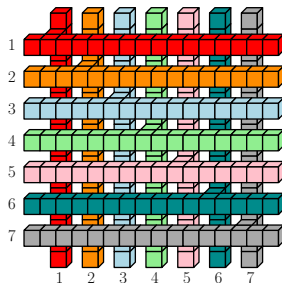
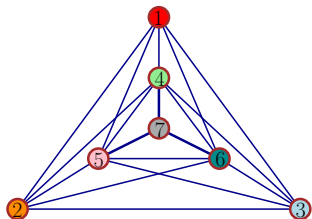


Voxel Representations for Graphs



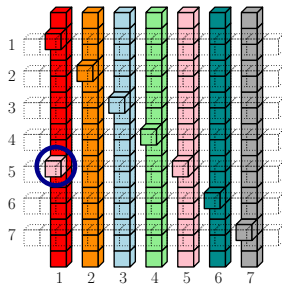
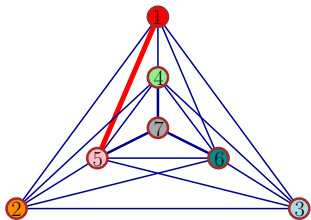
- Add diagonal voxels

Voxel Representations for Graphs



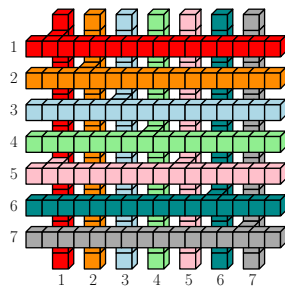
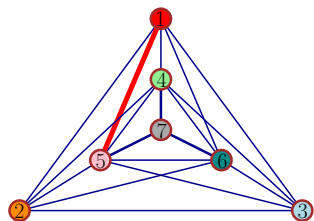
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Voxel Representations for Graphs



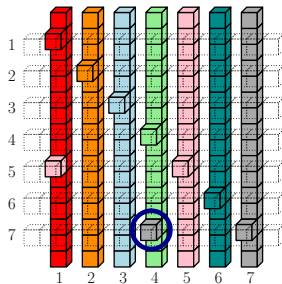
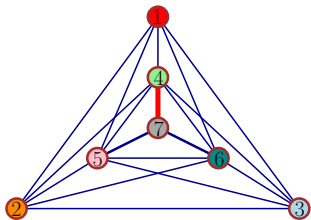
- Add diagonal voxels
- Add voxels for edges

Voxel Representations for Graphs



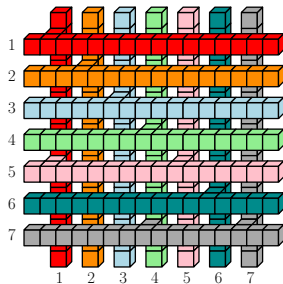
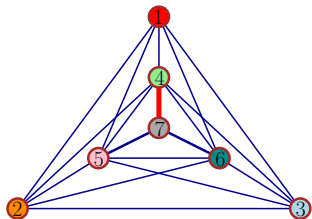
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Voxel Representations for Graphs



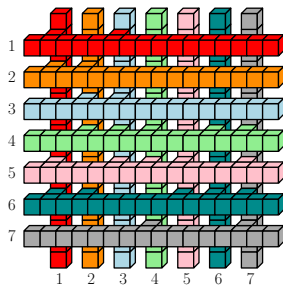
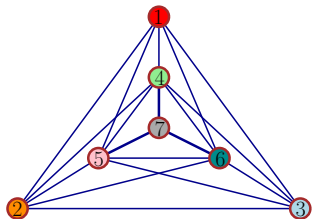
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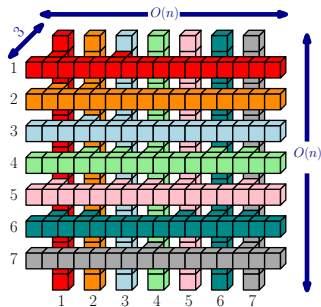
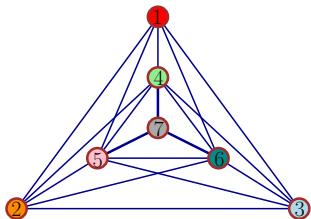
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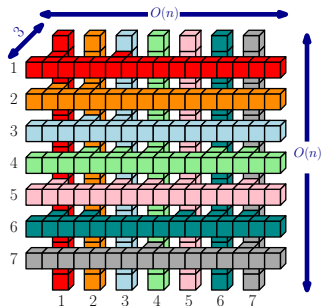
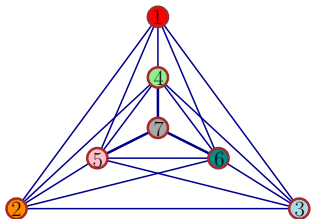


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Voxel Representations for Graphs



Voxel Representations for Graphs



Better bound for constant treewidth or constant genus

Our Result

Computational Complexity

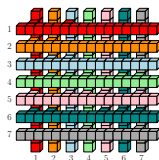
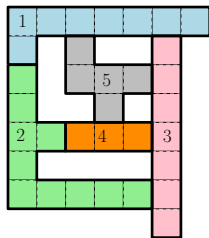
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Our Result

Computational Complexity

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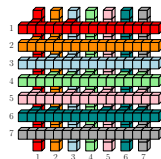
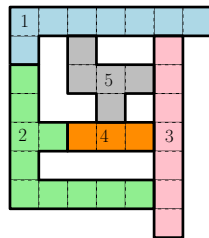
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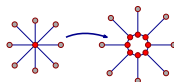
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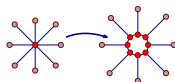
Voxel Representations for Genus- g Graphs

- Make the maximum degree 4

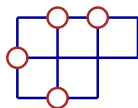


Voxel Representations for Genus- g Graphs

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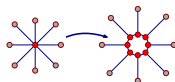


Orthogonal drawing on the plane (with crossing)
with total edge length $O((g + 1)^2 n \log^2 n)$ [Leiserson, 1980]



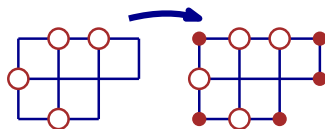
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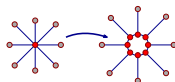
Orthogonal drawing on the plane (with crossing)
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- Subdivide at bend points



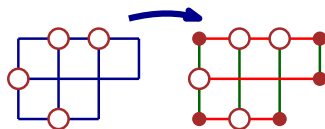
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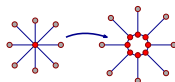
Orthogonal drawing on the plane (with crossing)
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- Subdivide at bend points
- Split horizontal and vertical graphs



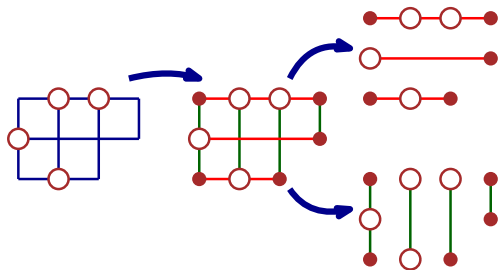
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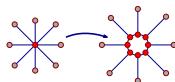
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- Split horizontal and vertical graphs



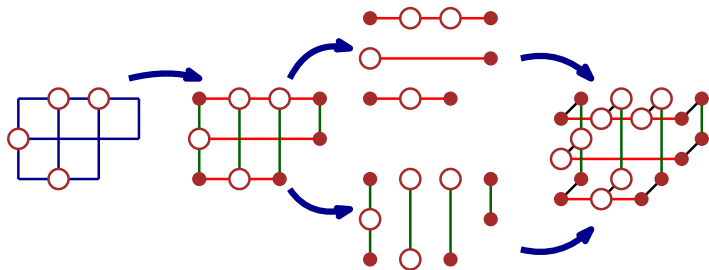
Voxel Representations for Genus- g Graphs

- Make the maximum degree 4



Orthogonal drawing on the plane (with crossing)
with total edge length $O((g+1)^2 n \log^2 n)$ [Leiserson, 1980]

- Subdivide at bend points
- Split horizontal and vertical graphs
- Combine horizontal and vertical graphs



Summary

Computational Complexity

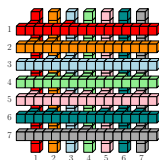
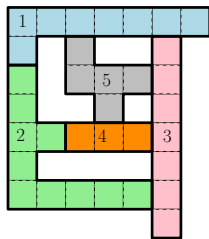
- Finding minimum-size representation is NP-complete in both 2D and 3D

Pixel Representation

- For a k -outerplanar graph, $\Theta(kn)$ pixels are necessary and sufficient

Voxel Representation

- $O(n^2)$ voxels are sufficient
- For a graph with treewidth τ , $\Theta(n \cdot \tau)$ voxels are necessary and sufficient
- For a graph with genus g , $O((g + 1)^2 n \log^2 n)$ voxels are sufficient



Future Work and Open Problems

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