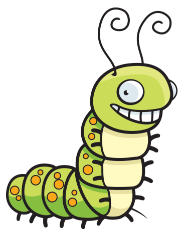
2-Layer Fan-planarity: From Caterpillar to Stegosaurus



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[2]Osnabrück University, Germany
[3]University of Cologne, Germany
[4]Monash University, Australia
[5]Charles University, Czech Republic
[6]University of Crete and FORTH, Greece

Thanks to BWGD 2015!

2-Layer Drawings: Definition

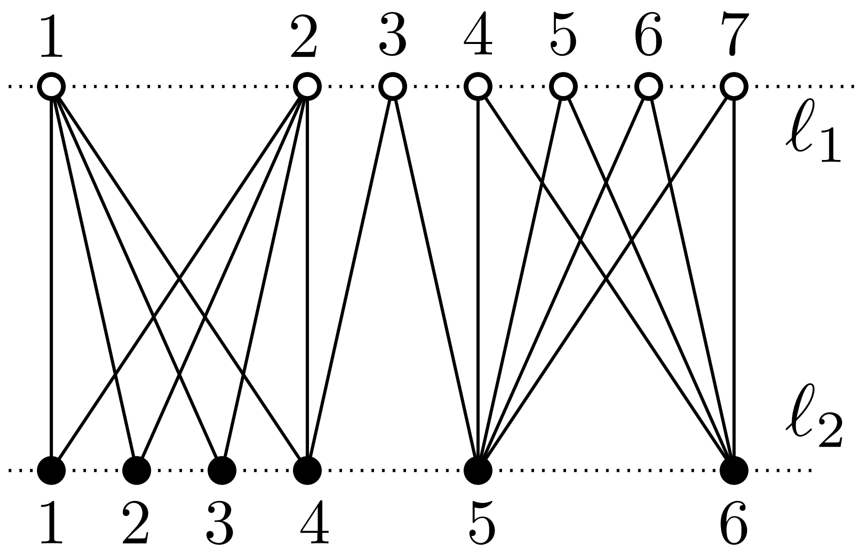
2-layer drawing of a graph:

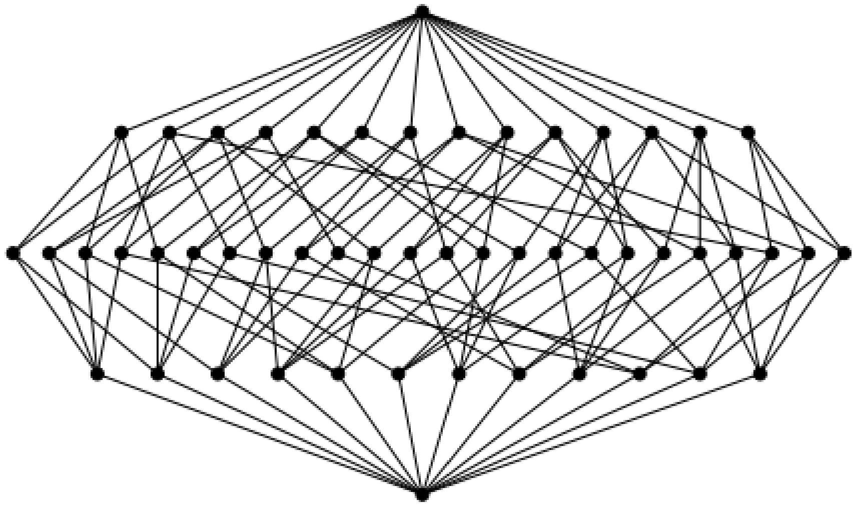
- each vertex is a point of one of two horizontal layers
- each edge is a straight-line segment that connects vertices of different layers

Fact: G has a 2-layer drawing if and only if is bipartite

Motivation:

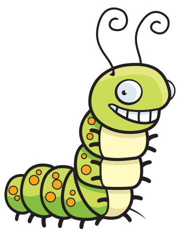
- convey bipartite graphs
- building block of layered drawings

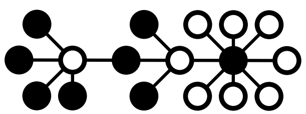


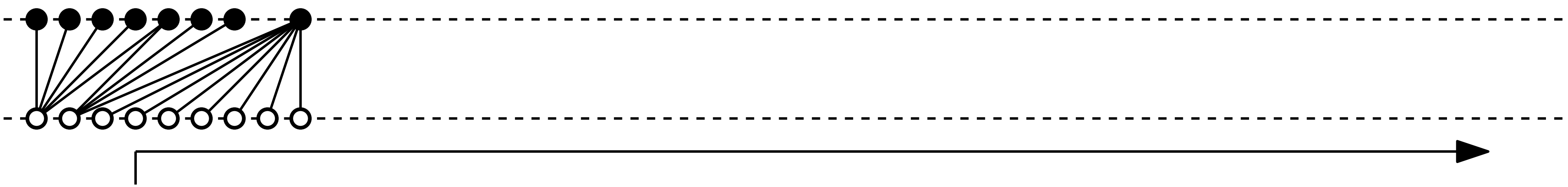


2-Layer Drawings: Evolution

PLANARITY
AGE







1986

Name: Caterpillar

Family: Planar

Eades *et al.*

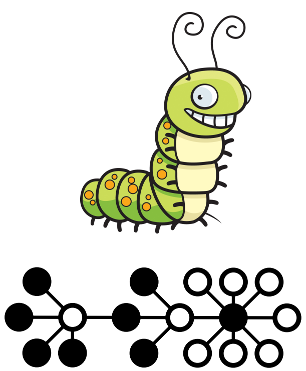
2-Layer Drawings: Evolution

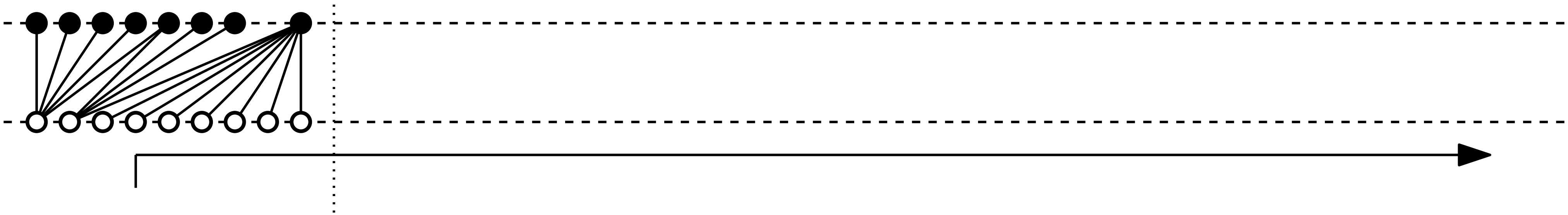
PLANARITY
AGE

Non-planar drawings: minimizing the number of crossing edges in a 2-layer drawing in NP-hard [Eades and Whitesides, 1994]

Subsequent papers:

- heuristics for crossing minimization
- restrictions on crossings (this paper)





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PLANARITY AGE

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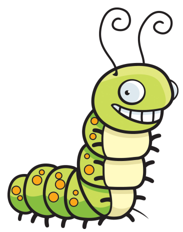

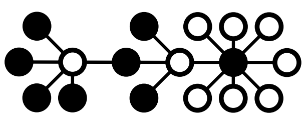

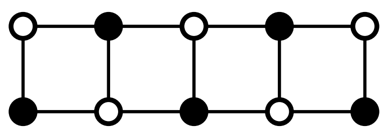

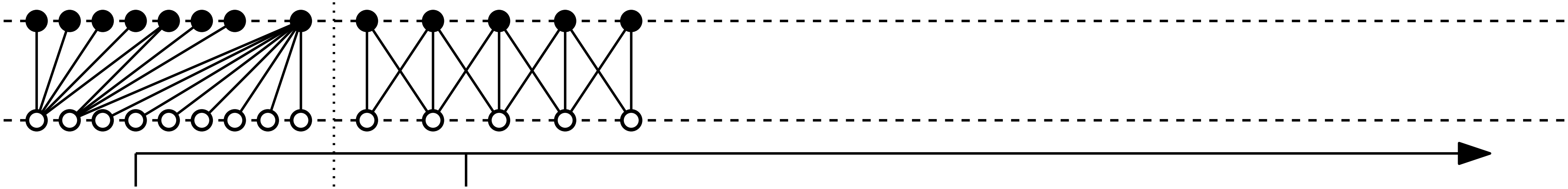

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PLANARITY AGE

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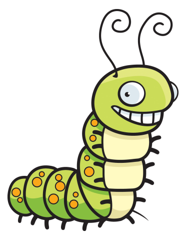

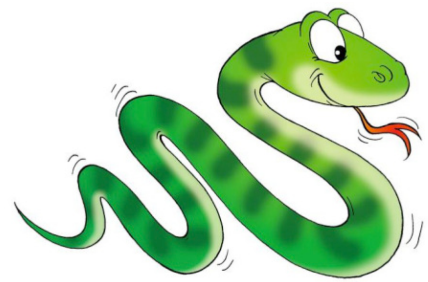

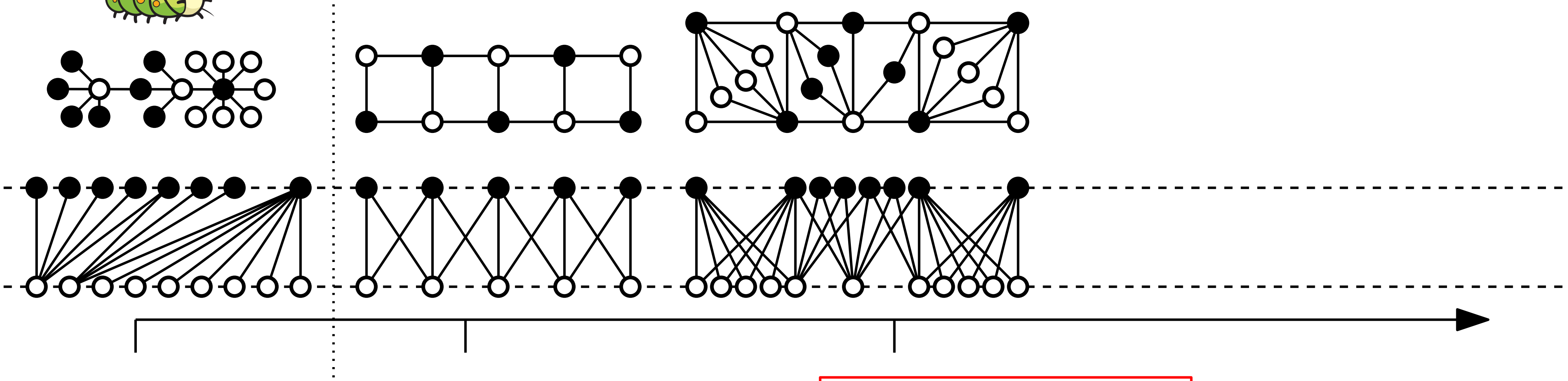

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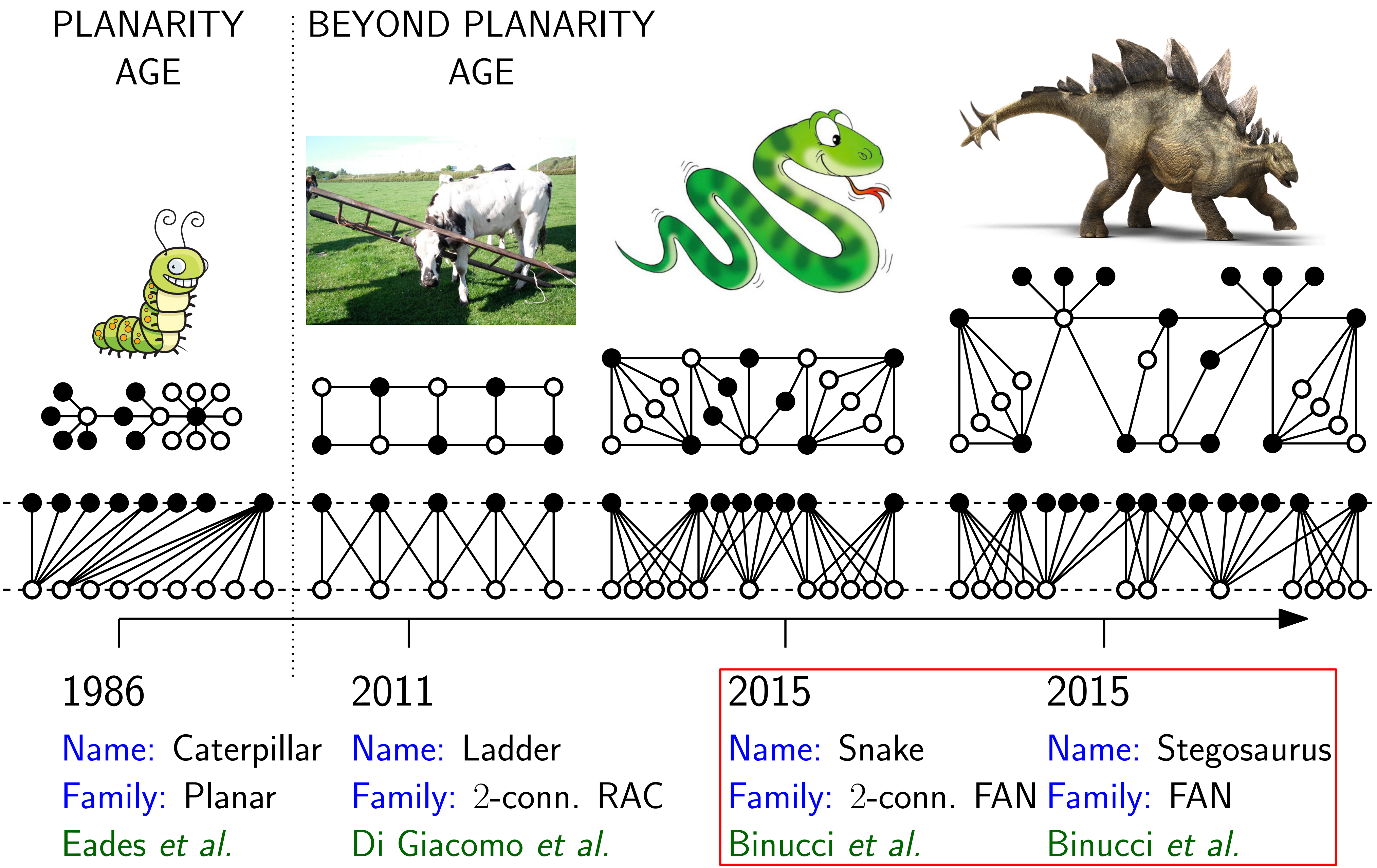
2015

Name: Snake

Family: 2-conn. FAN

Binucci *et al.*

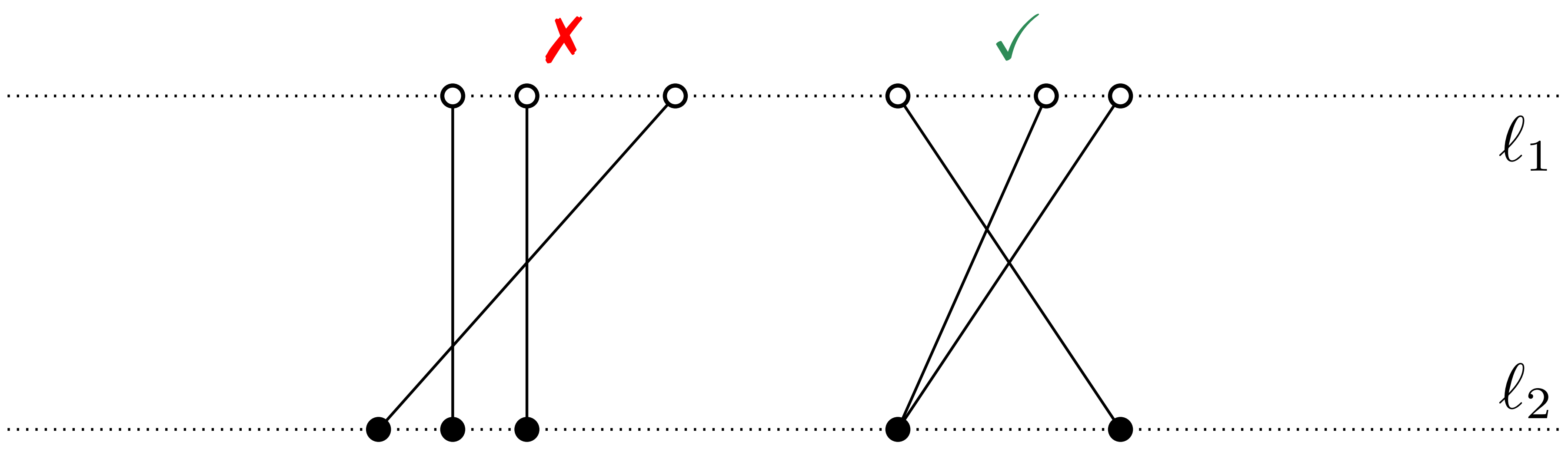
2-Layer Drawings: Evolution



2-Layer Fan-planar Drawings: Definition

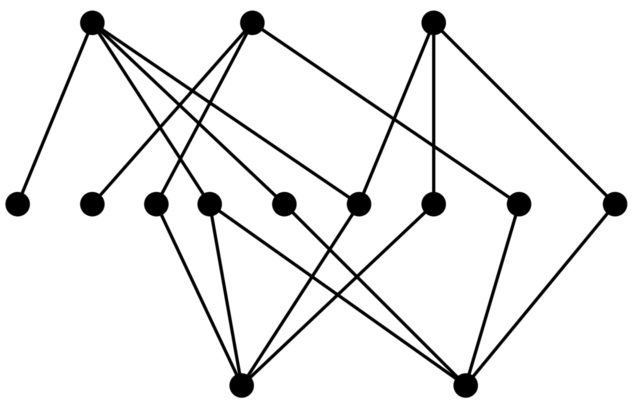
A drawing is *fan-planar* if there is no edge that crosses two other independent edges [Bekos *et al.*, 2014; Binucci *et al.*, 2014; Kaufmann and Ueckerdt, 2014]

A *2-layer fan-planar drawing* is a 2-layer drawing that is also fan-planar.

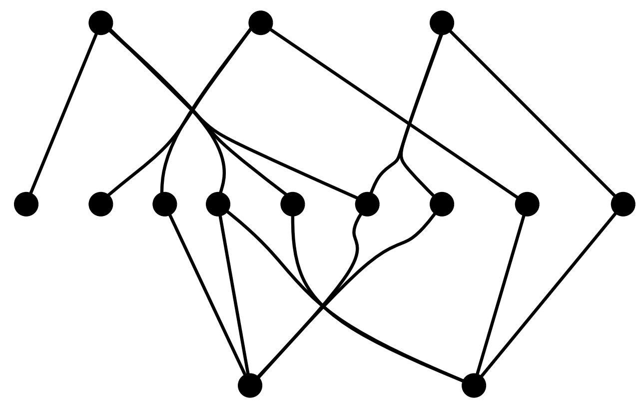


2-Layer Fan-planar Drawings: Application

Application: they can be used as a basis for generating drawings with few edge crossings in a confluent drawing style [Dickerson *et al.*, 2005; Eppstein *et al.*, 2007]

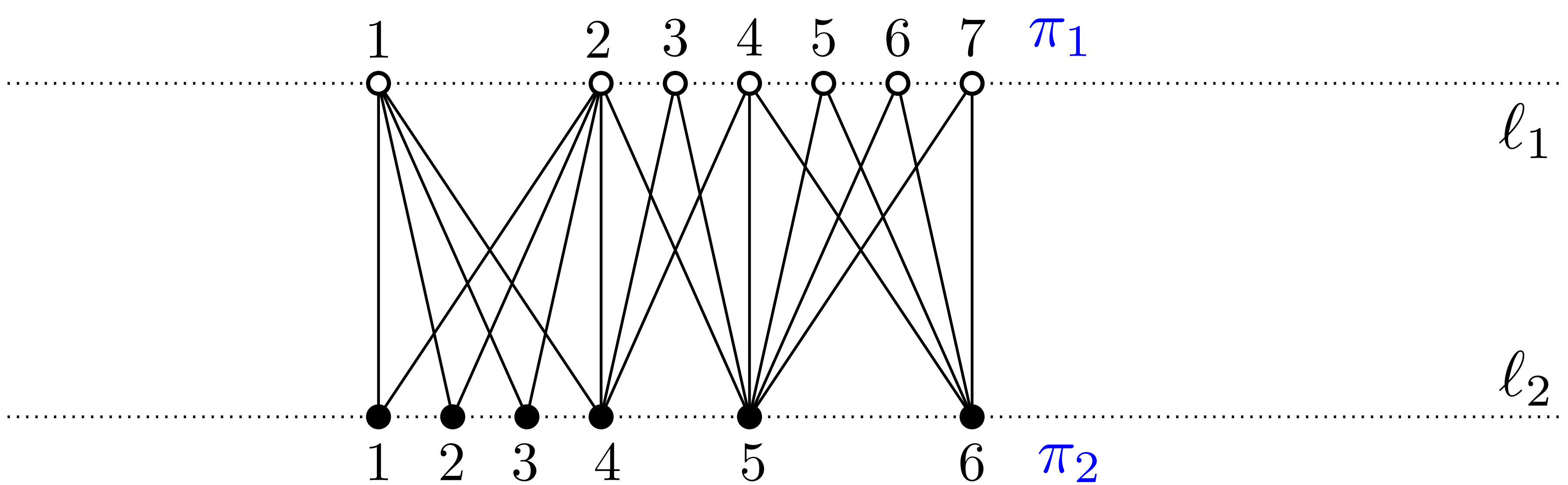


better readability



2-Layer Fan-planar Drawings: Notation

A *2-layer embedding* is an equivalence class of 2-layer drawings, described by a pair of linear orderings $\gamma = (\pi_1, \pi_2)$



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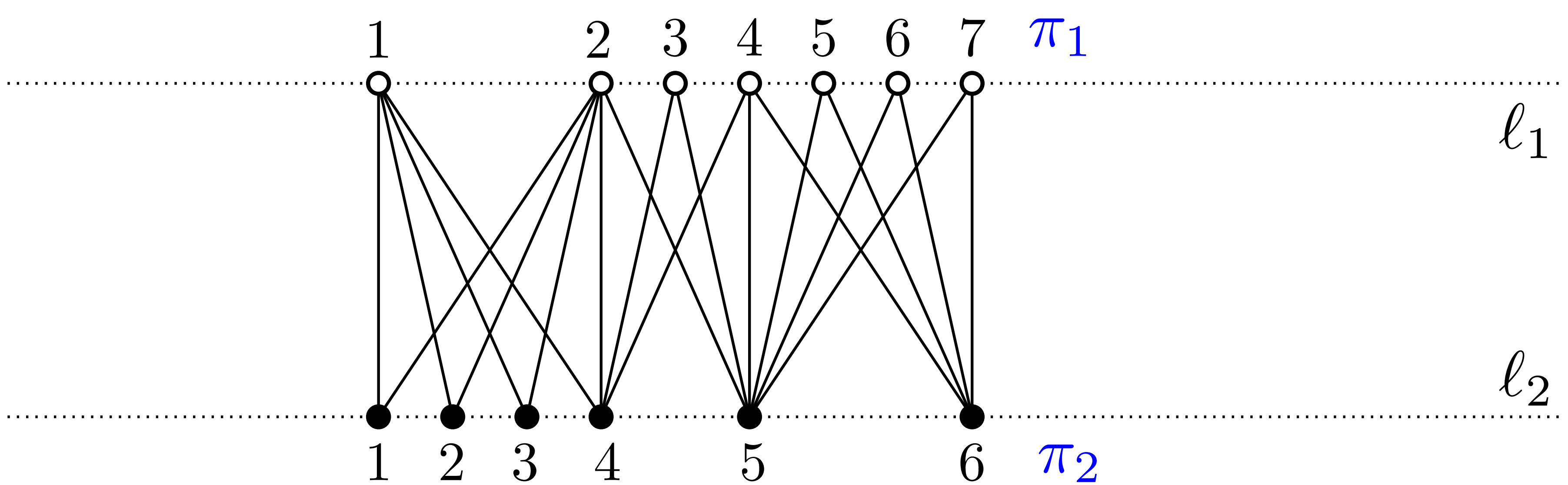

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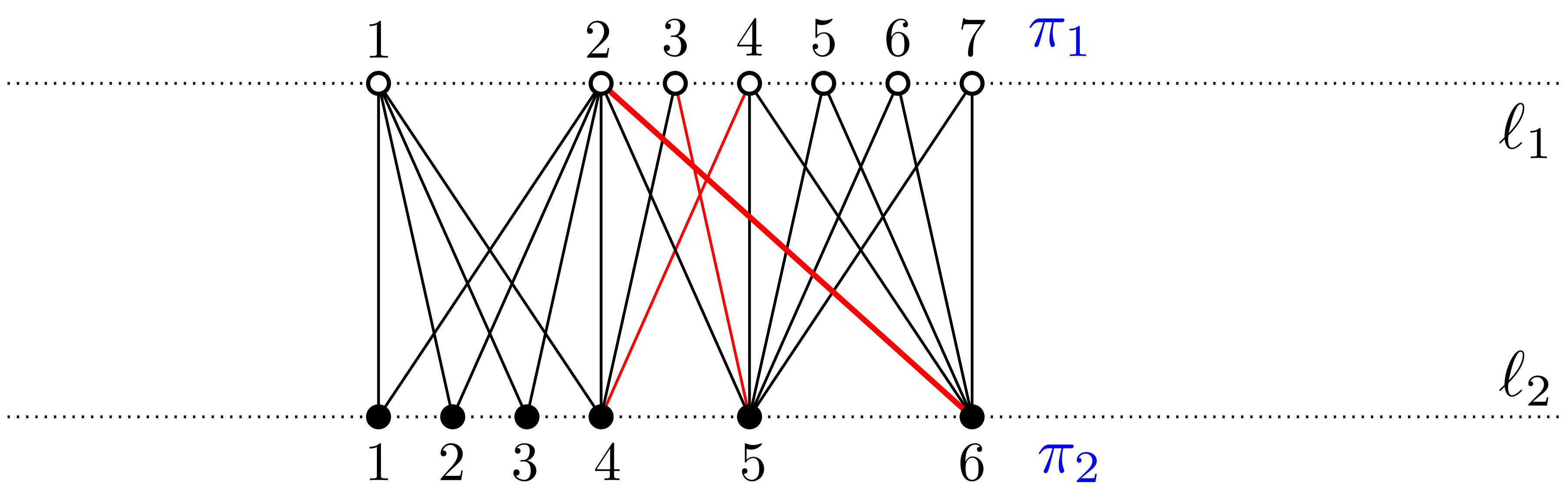

Characterization of biconnected 2-layer fan-planar graphs

Snake: Definition

Definition 1. A *snake* is recursively defined as follows:

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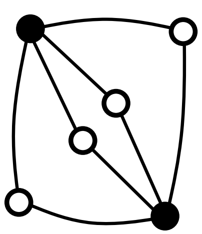
$K_{2,4}$



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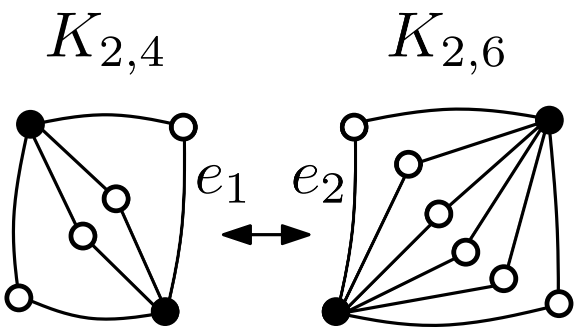
  A vertex can be merged just once!



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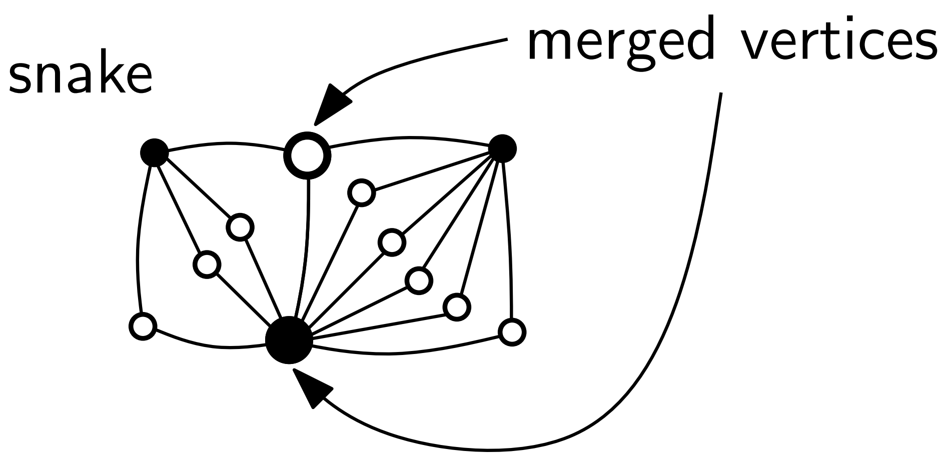

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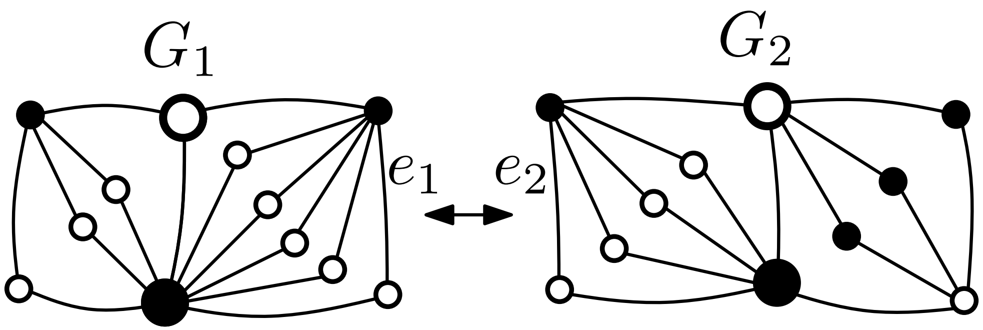

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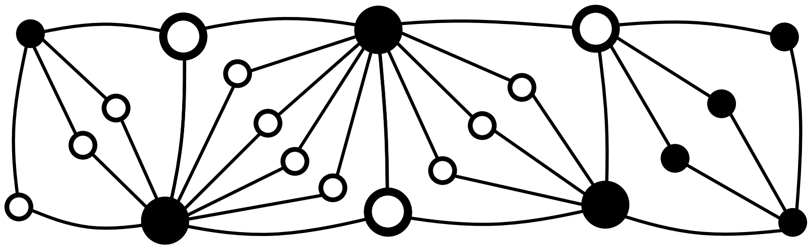
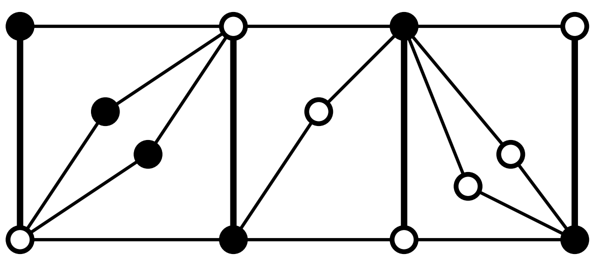


2-Layer Bicon. Fan-planar Graph $\leftarrow$ Snake

Lemma 1 *Every n-vertex snake admits a 2-layer fan-planar embedding, which can be computed in $O(n)$ time.*



ℓ_1

ℓ_2

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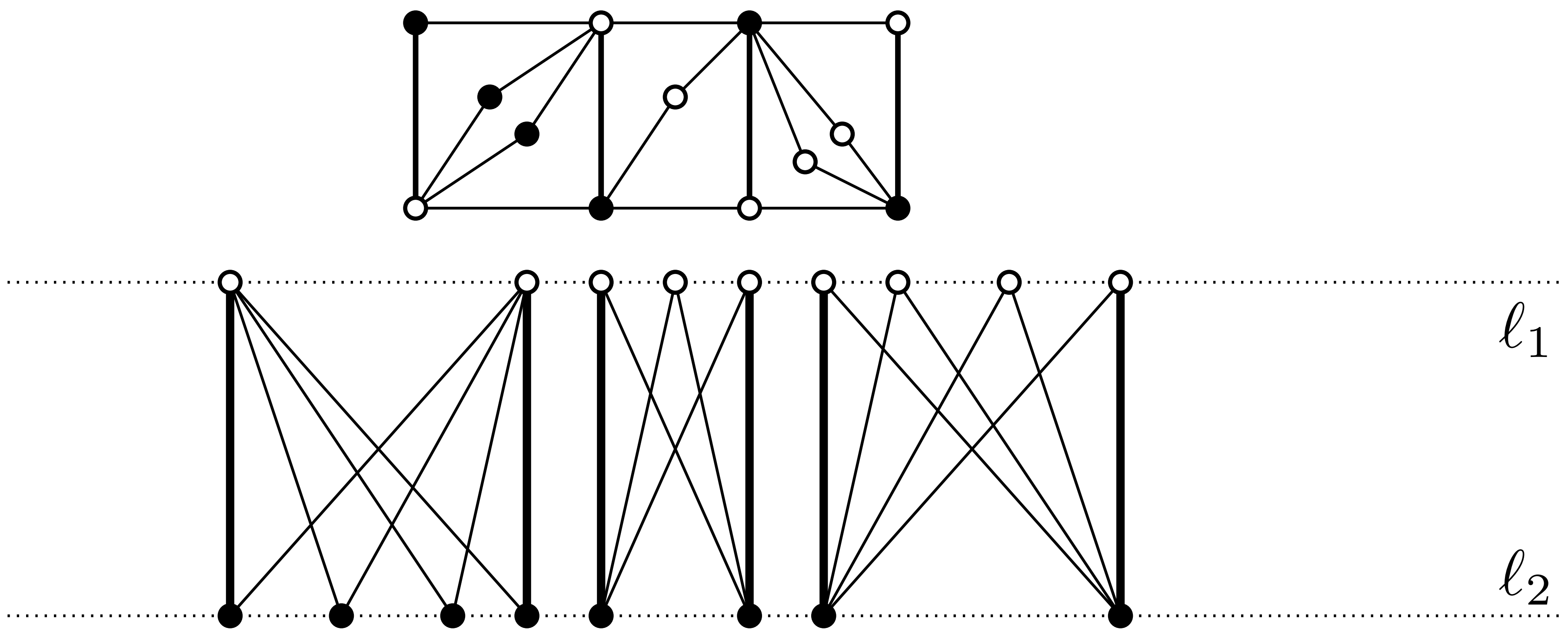
Idea:

- Draw each $K_{2,h}$ independently



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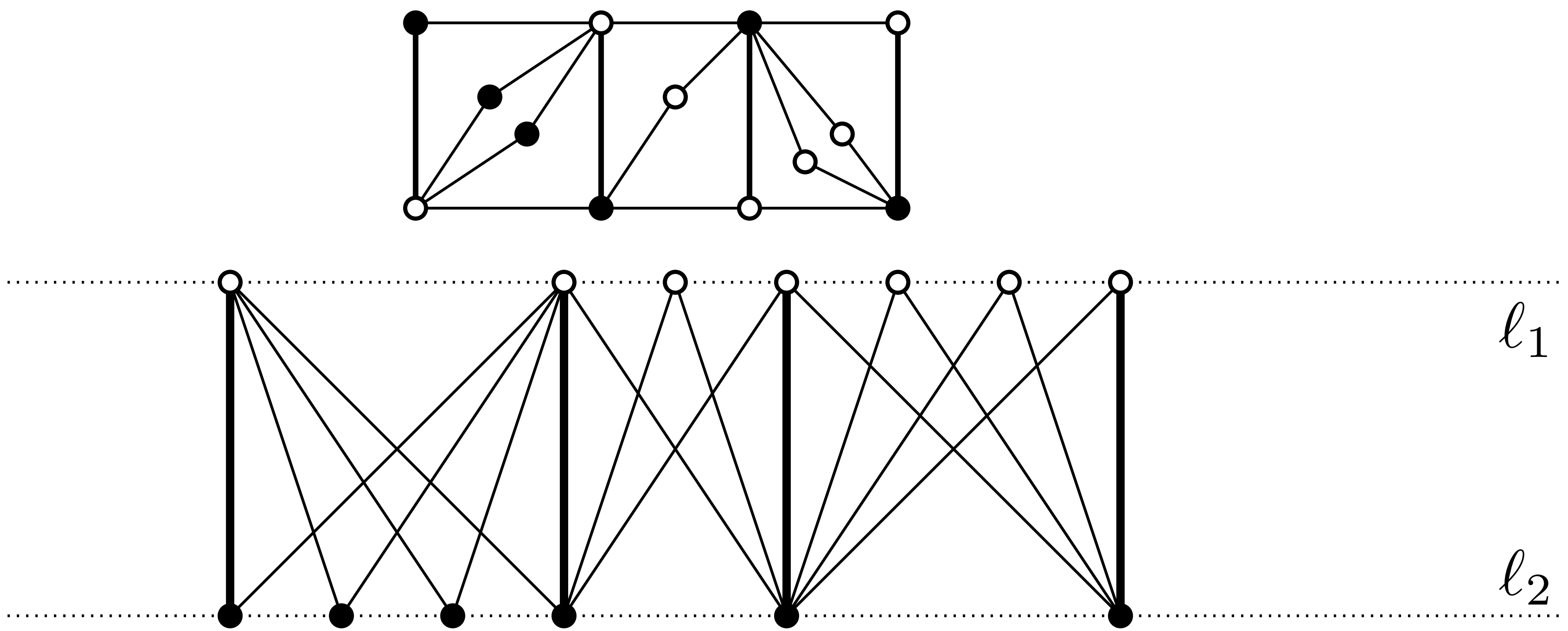
- Merge the drawings



2-Layer Bicon. Fan-planar Graph $\rightarrow$ Snake

Lemma 2 *Let* G *be biconnected graph. If* G *admits a maximal* 2*-layer fan-planar embedding* γ *then* G *is a snake.*

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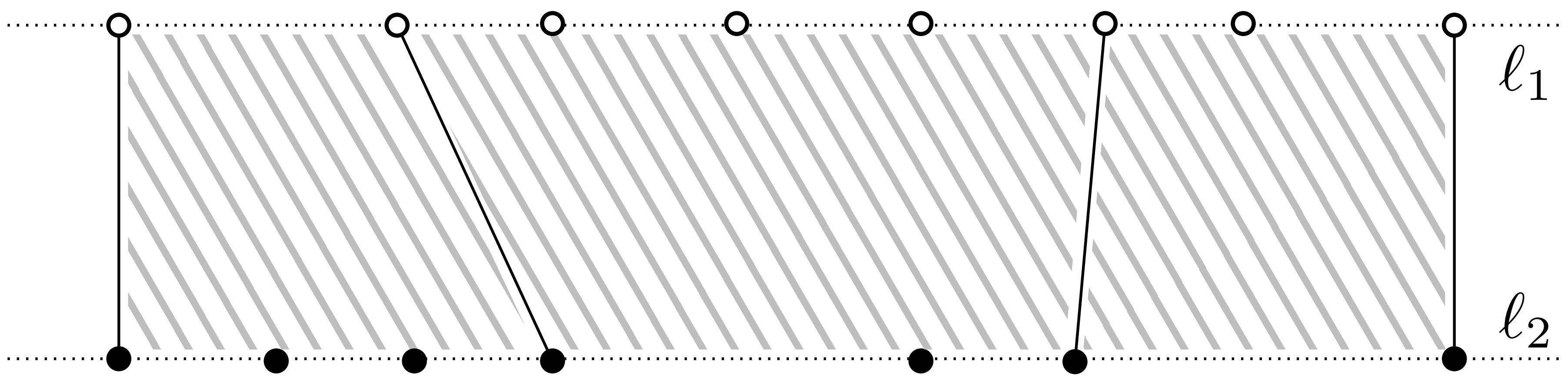
Lemma 2 *Let G be biconnected graph. If G admits a maximal 2-layer fan-planar embedding γ then G is a snake.*

Idea: Decompose γ by "splitting" the uncrossed edges



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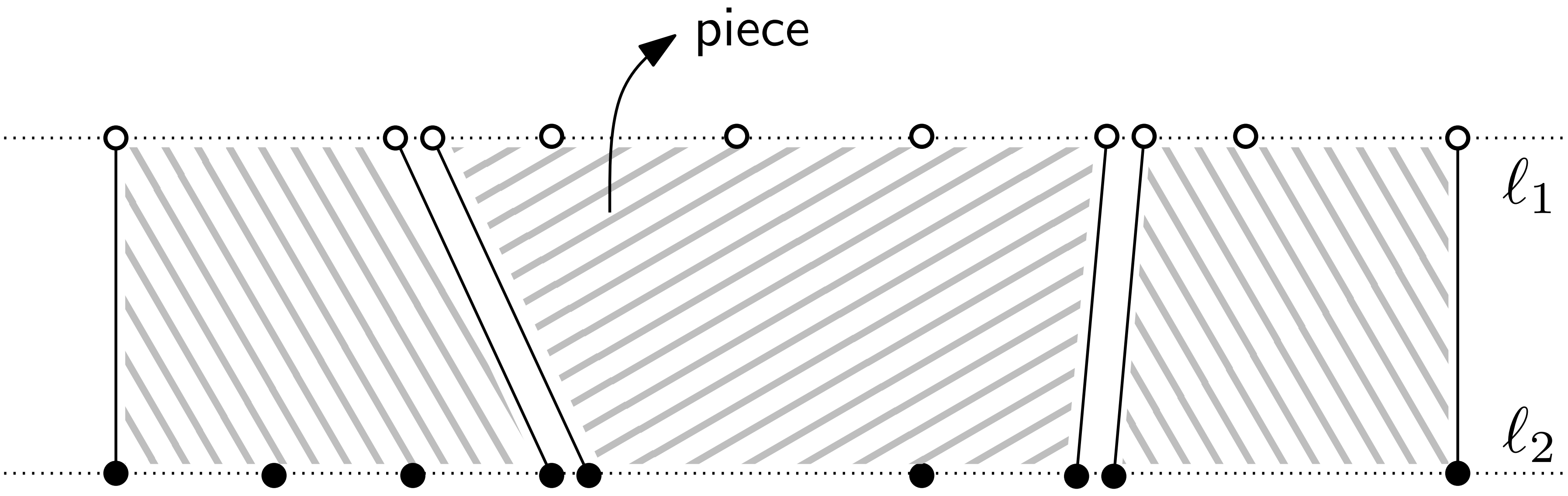
Prove that each piece is a $K_{2,n}$ (for some $n \geq 2$)



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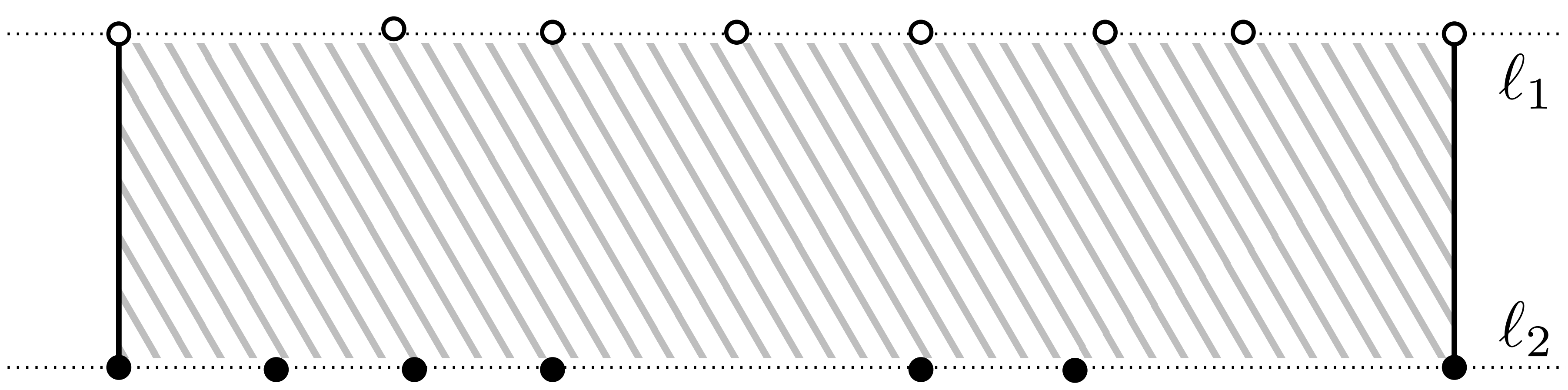

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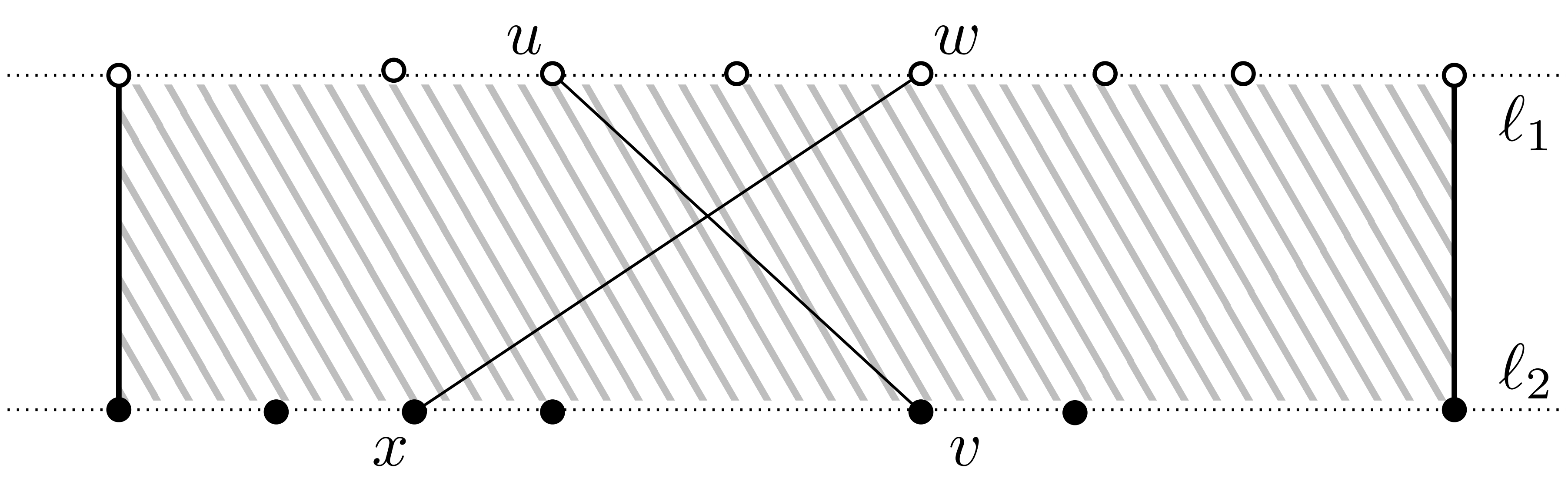

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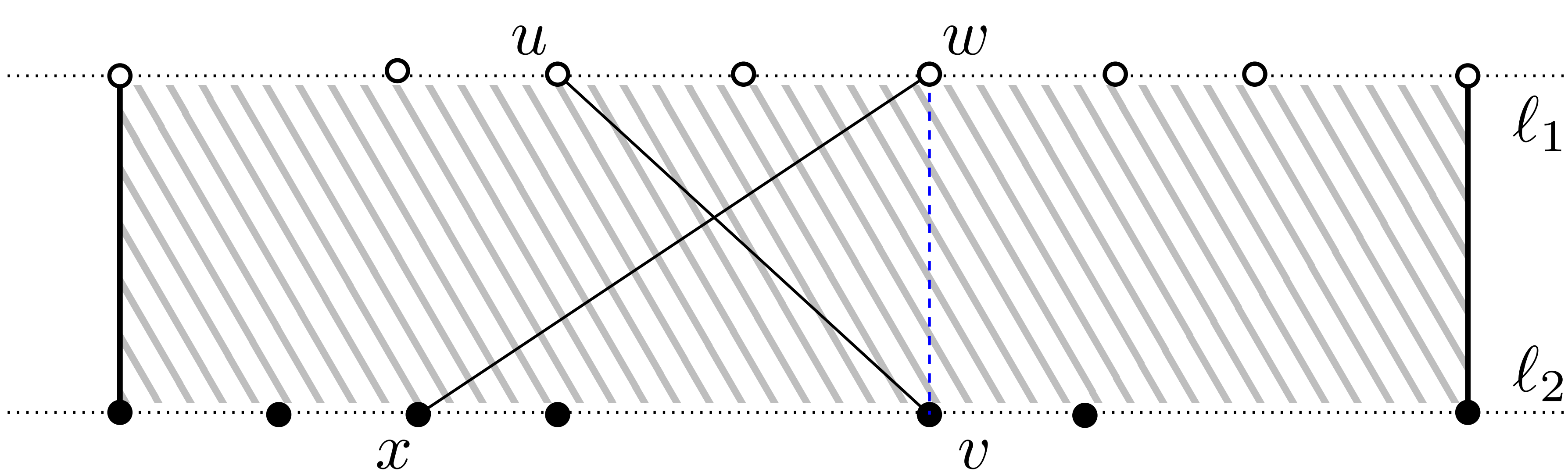
Consider the segment $\overline{wv}$:

Case 1: No edge traverses $\overline{wv}$



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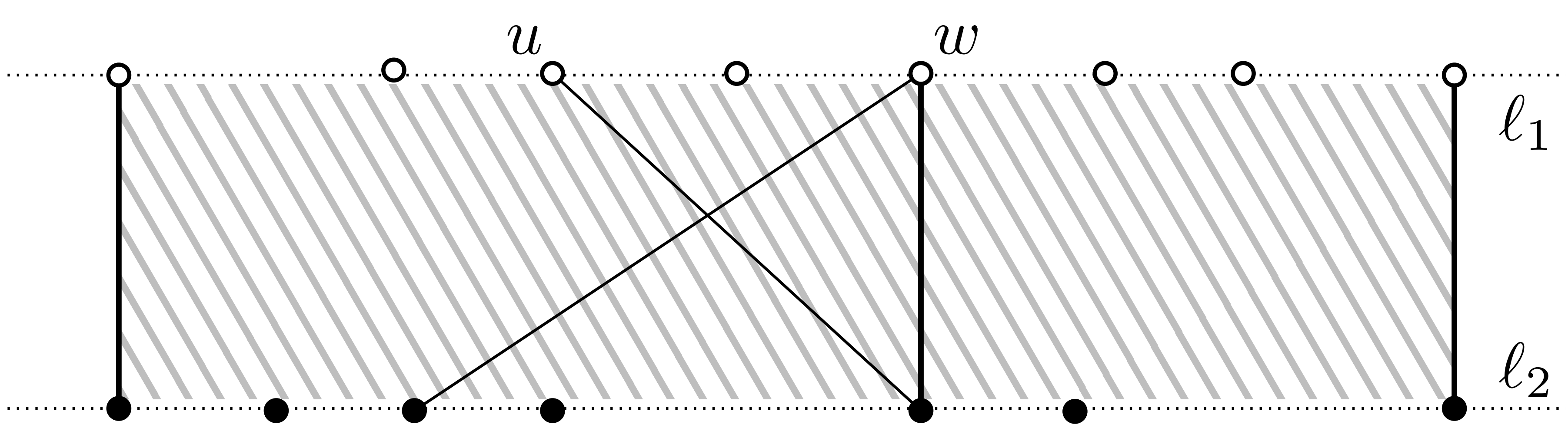
Then due to maximality (w, v) exists



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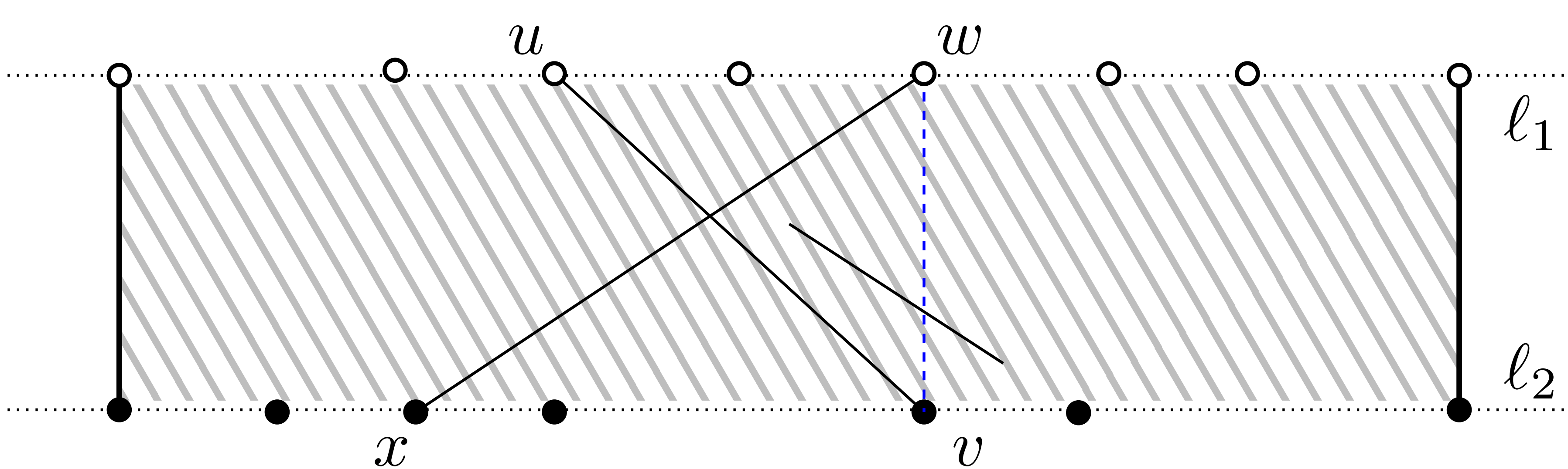

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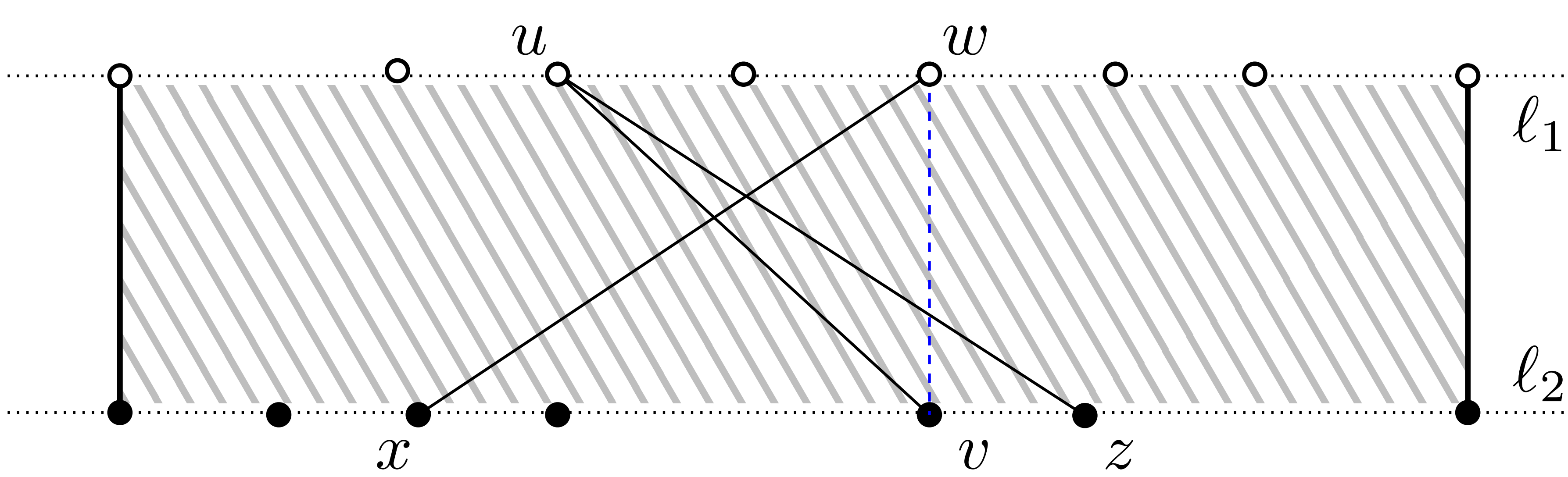
Due to fan-planarity, one end-vertex of e must be either u or x



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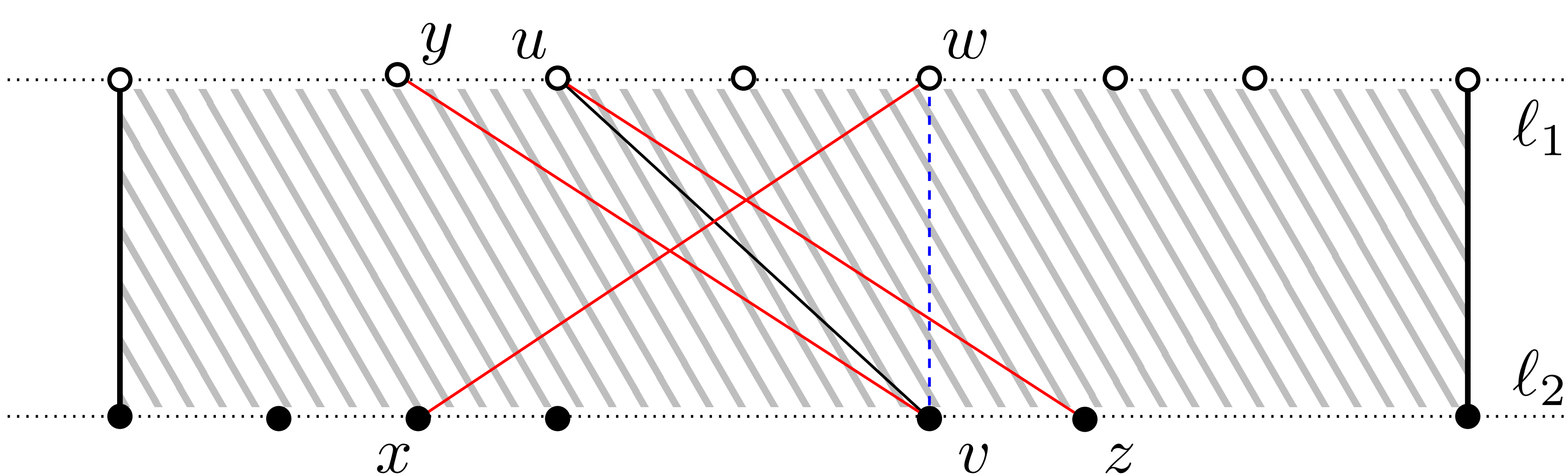
Any edge (y,v) is s.t. $y = w$, otherwise γ is not fan-planar



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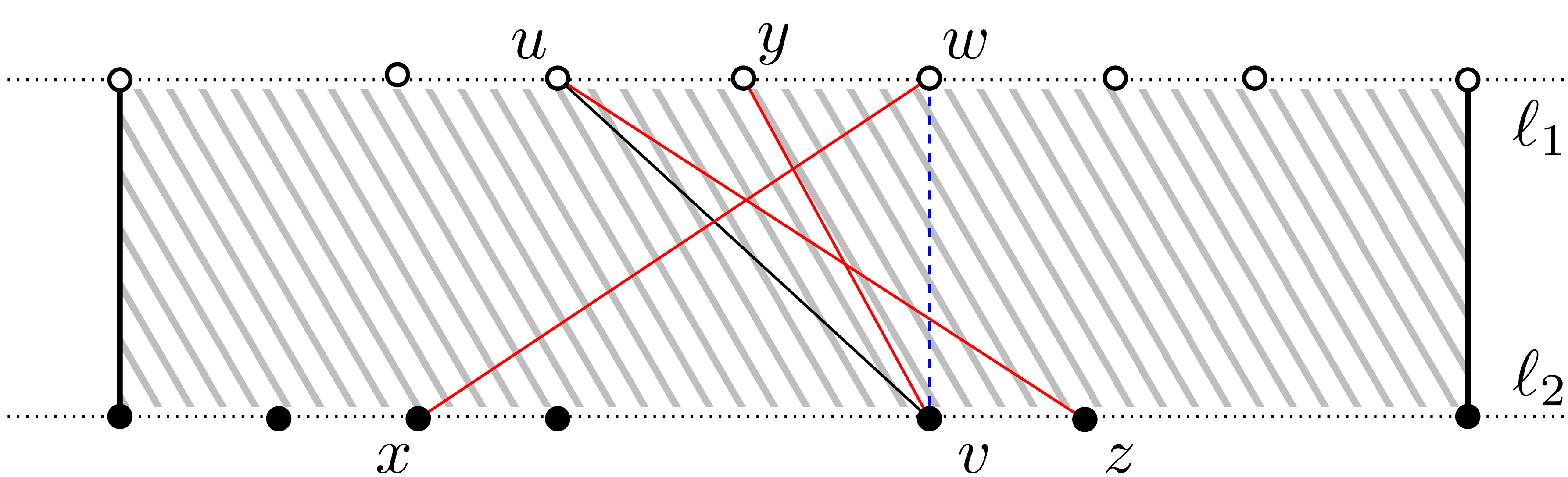

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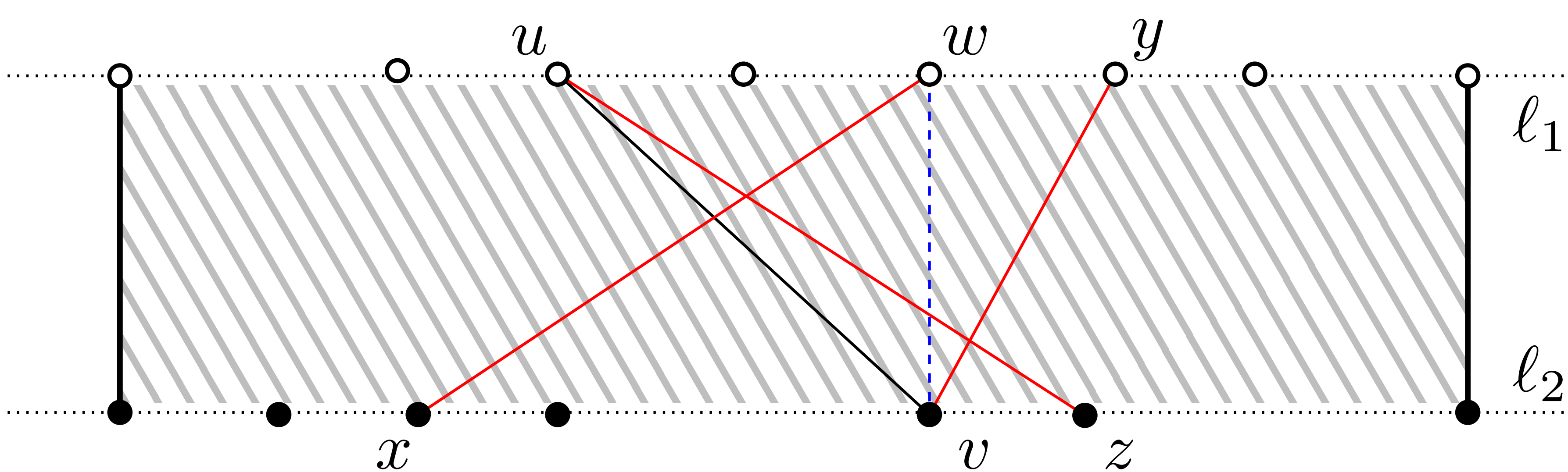

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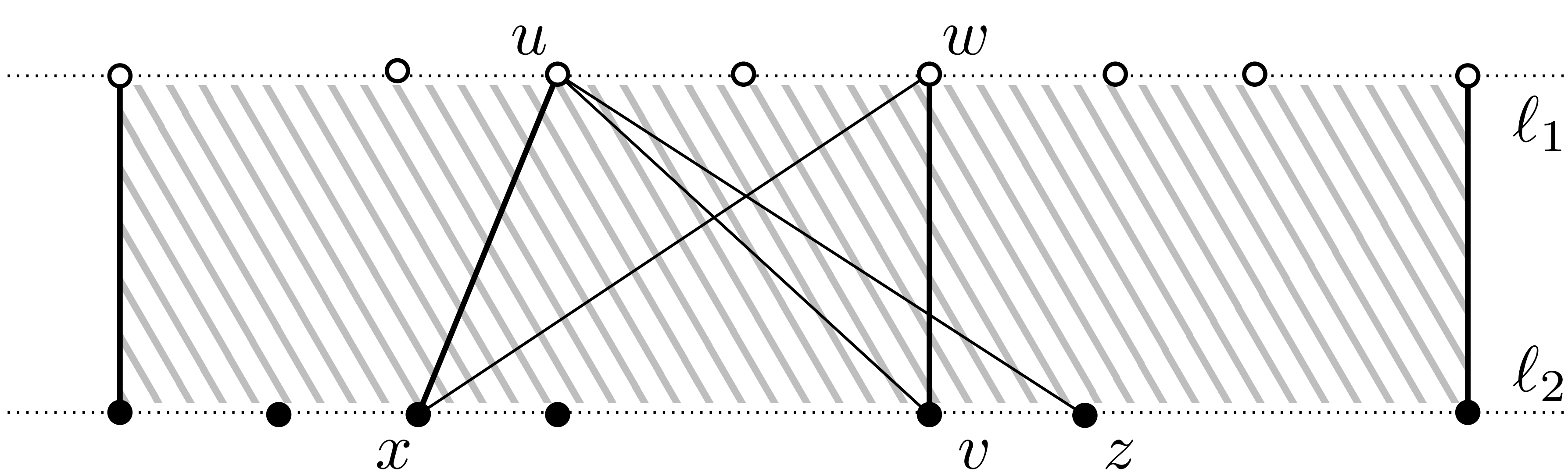

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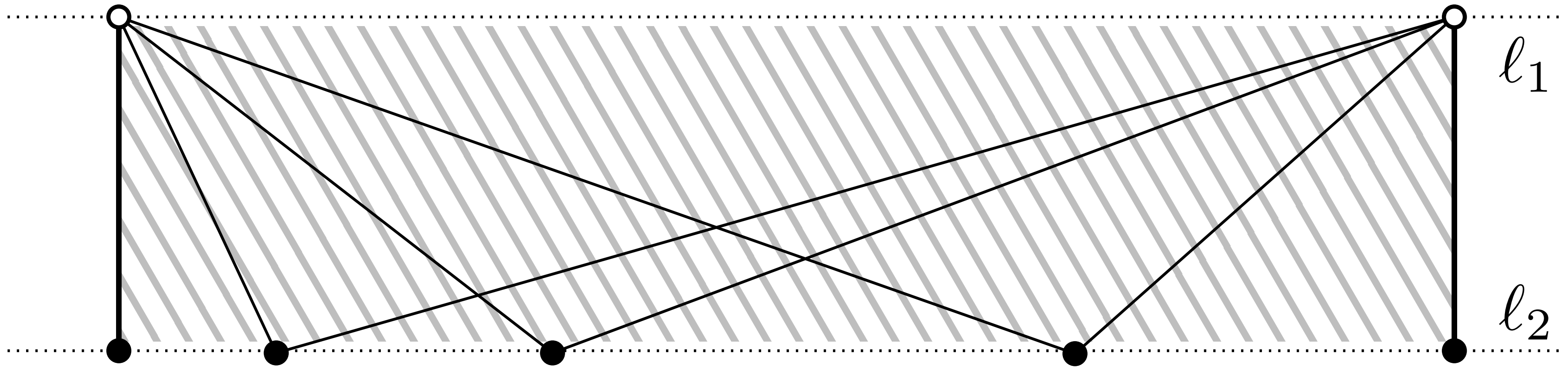

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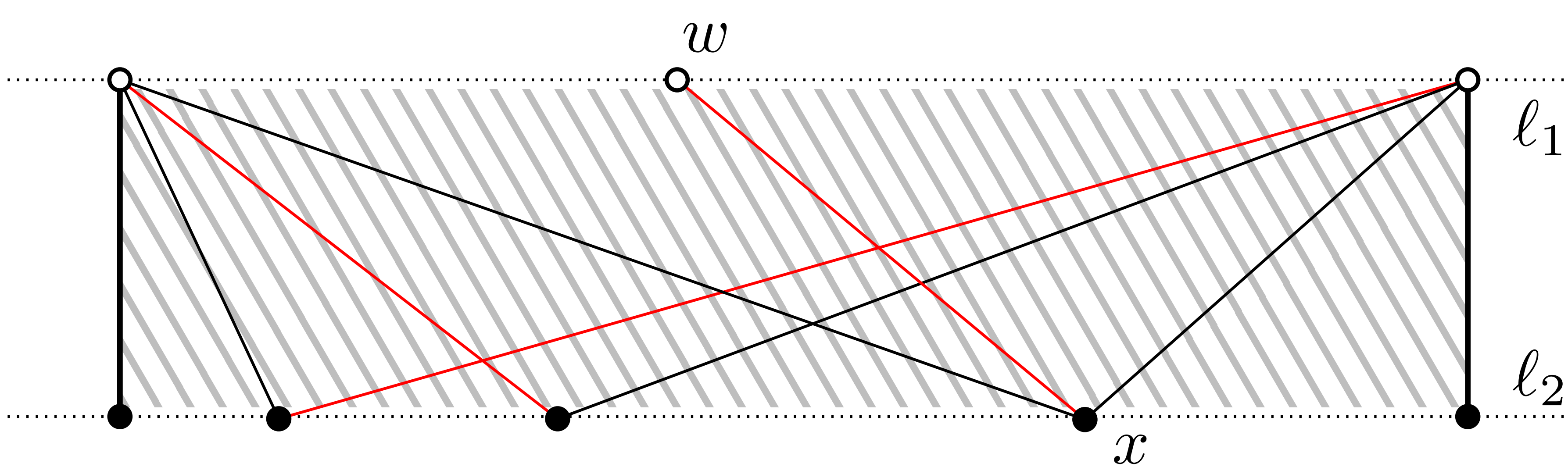
Suppose there is another vertex w on ℓ_1

Any edge (w, x) would violate fan-planarity

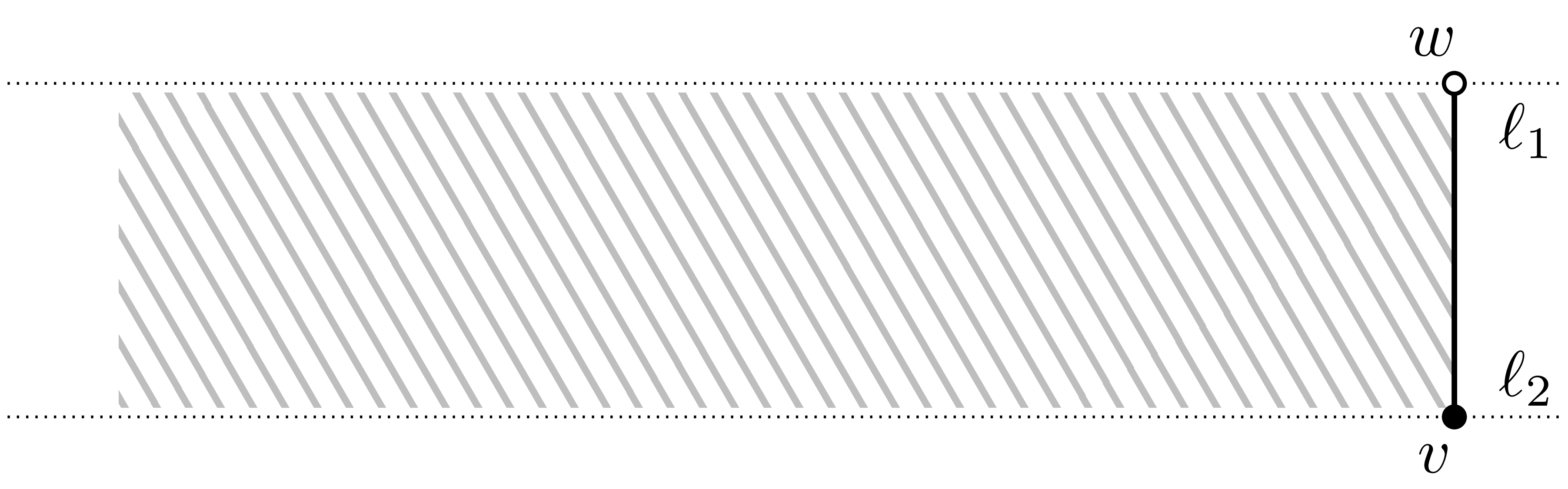


2-Layer Bicon. Fan-planar Graph $\rightarrow$ Snake

Lemma 2 *Let G be biconnected graph. If G admits a maximal 2-layer fan-planar embedding γ then G is a snake.*

We prove that each piece P is a $K_{2,n}$ for some $n \geq 2$.

Consider now the rightmost vertices of $\gamma[P]$.



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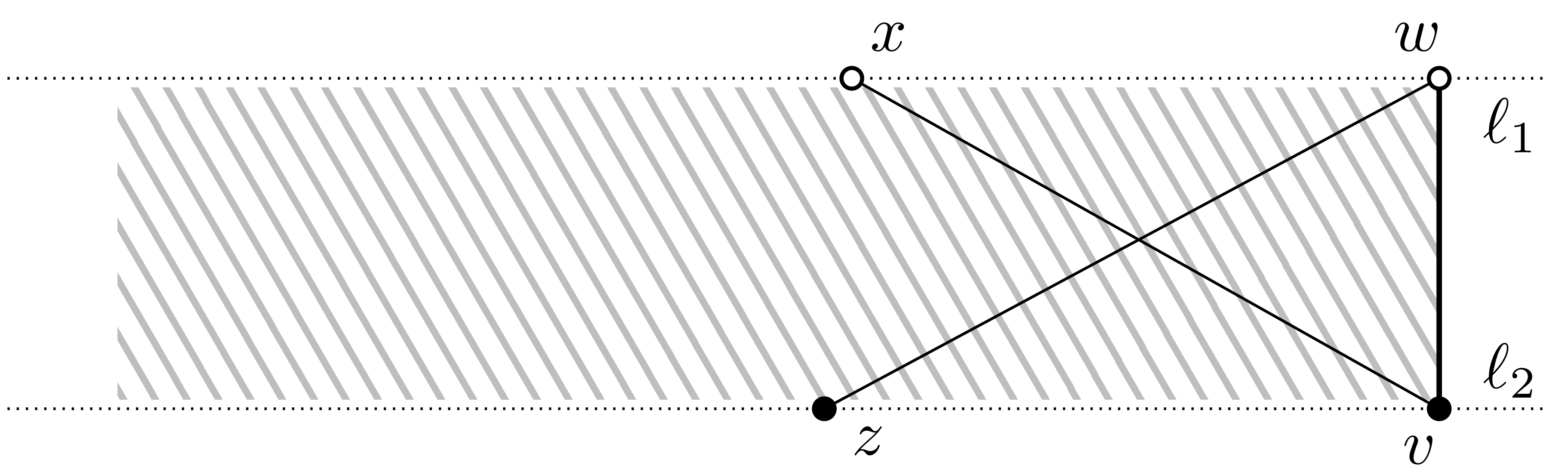

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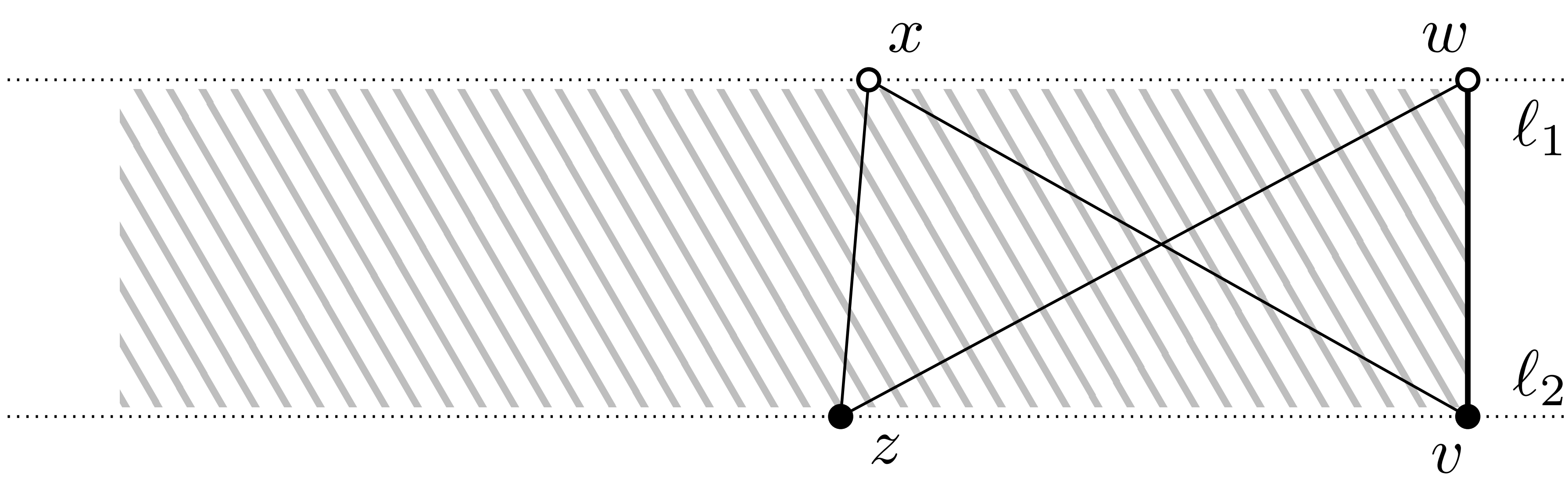

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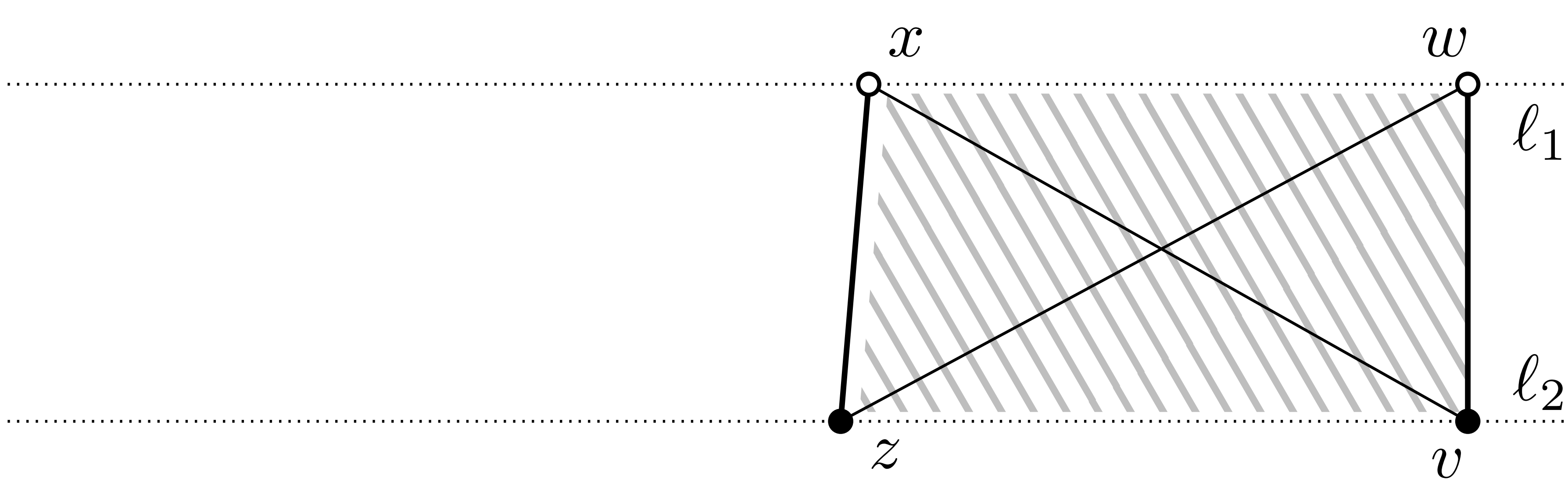
If (x, z) is uncrossed, by Claim 2 the statemente follows.



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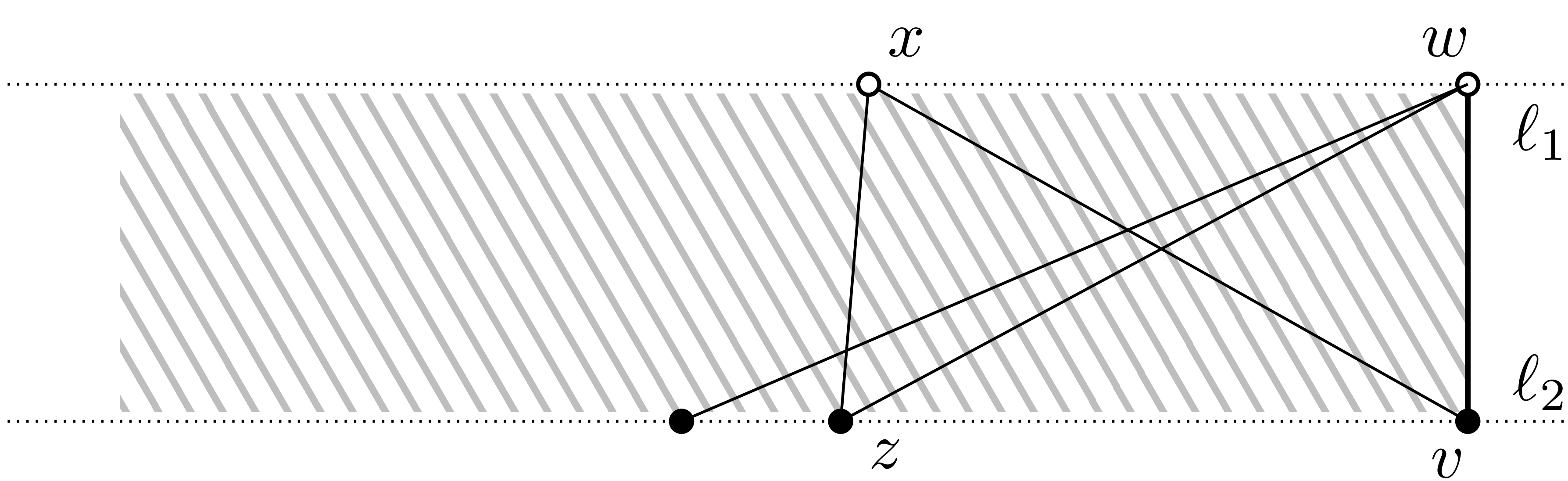
Otherwise it is crossed by an edge having w or v as an end-vertex...



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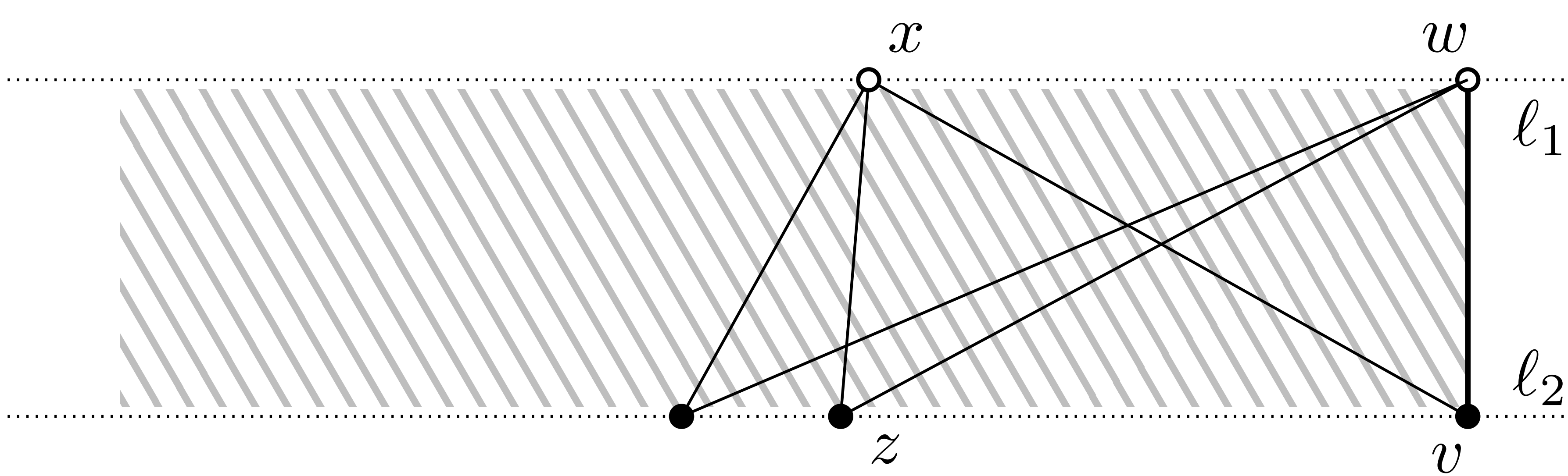
Iterate until we hit the leftmost uncrossed edge of P (and then apply Claim 2)



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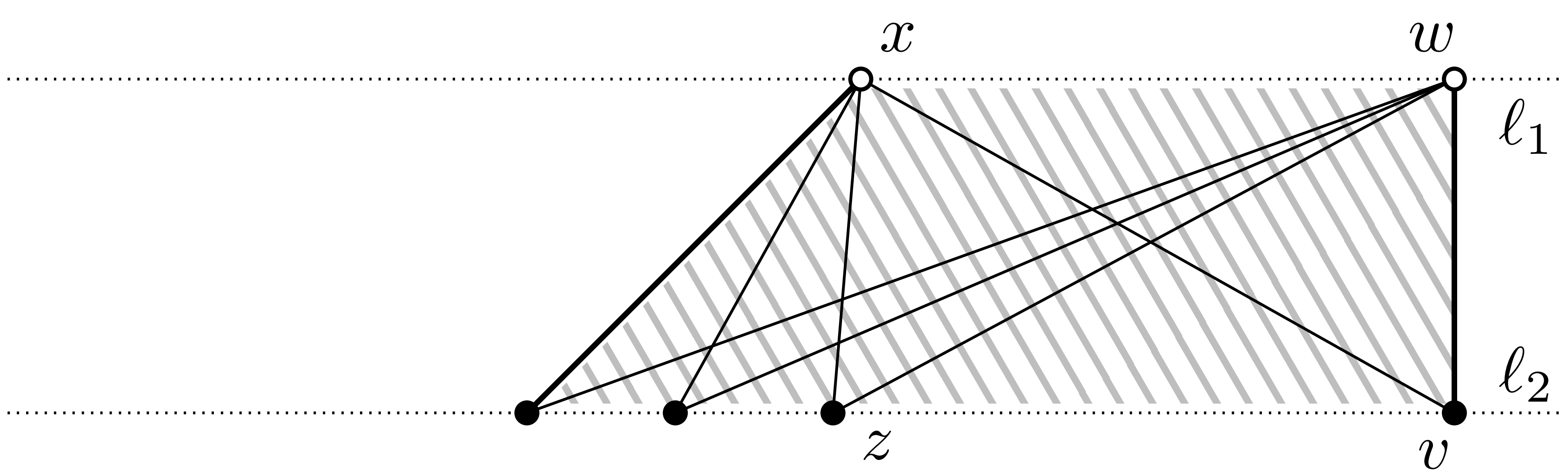

2-Layer Bicon. Fan-planar Graph $\iff$ Snake

Theorem 1 *A biconnected graph G is 2-layer fan-planar if and only if G is a spanning subgraph of a snake.*

Lemma 1 $+$ Lemma 2.

Testing biconnected graphs

Test for biconnected graphs

Theorem 2 *Let G be a bipartite biconnected graph with n vertices. There exists an $O(n)$-time algorithm that tests whether G is 2-layer fan-planar, and that computes a 2-layer fan-planar embedding of G in the positive case.*

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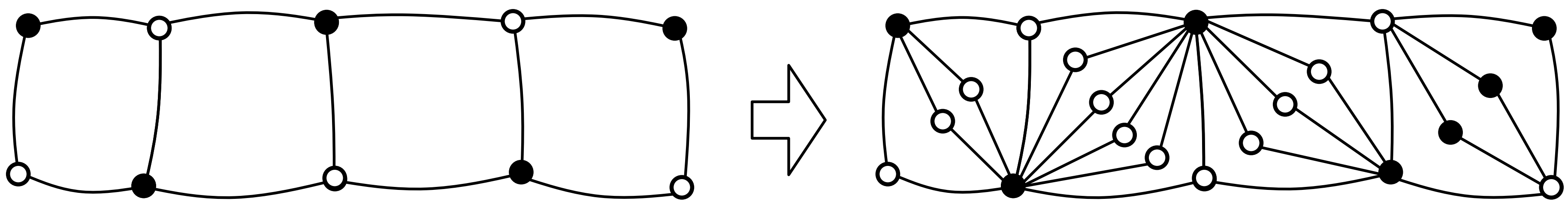
Idea: Check if G can be augmented to a snake by adding only edges.

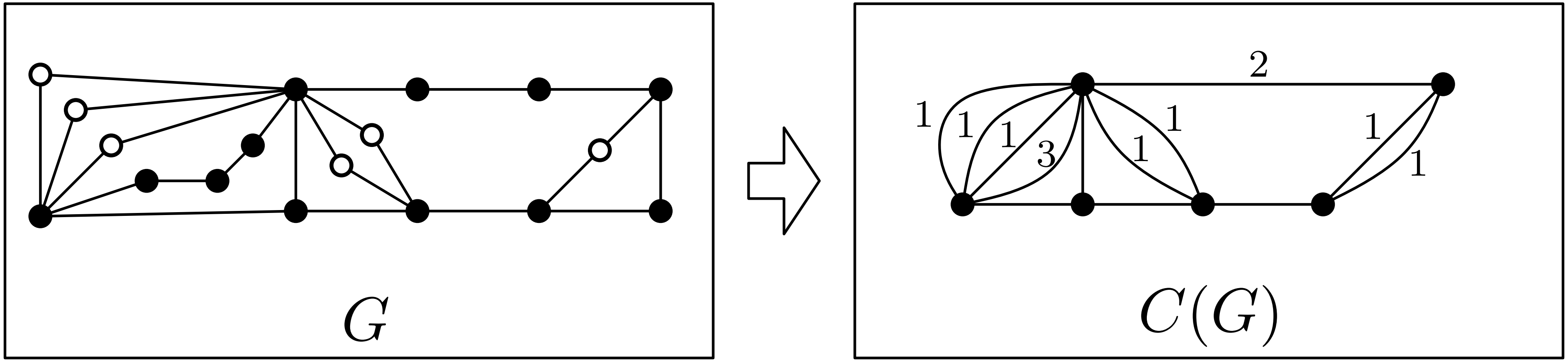
Observation: snake $=$ ladder $+$ paths of length 2 inside inner faces



Test for biconnected graphs: Algorithm

Step 1: Contract each chain into a weighted edge.
Construct (if any) an outerplanar embedding of the graph.

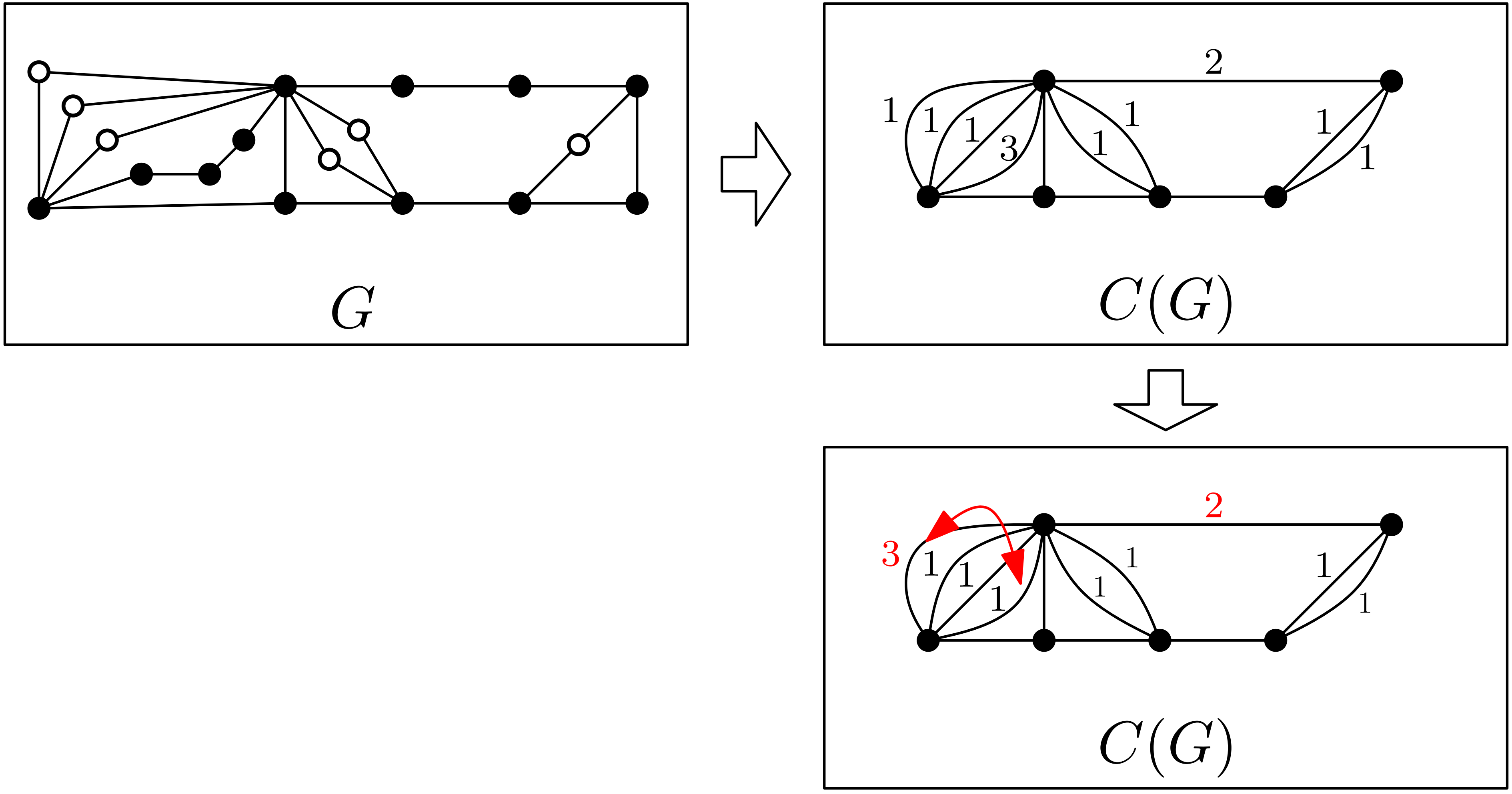
Observation: Inner paths all have weight 1.



Test for biconnected graphs: Algorithm

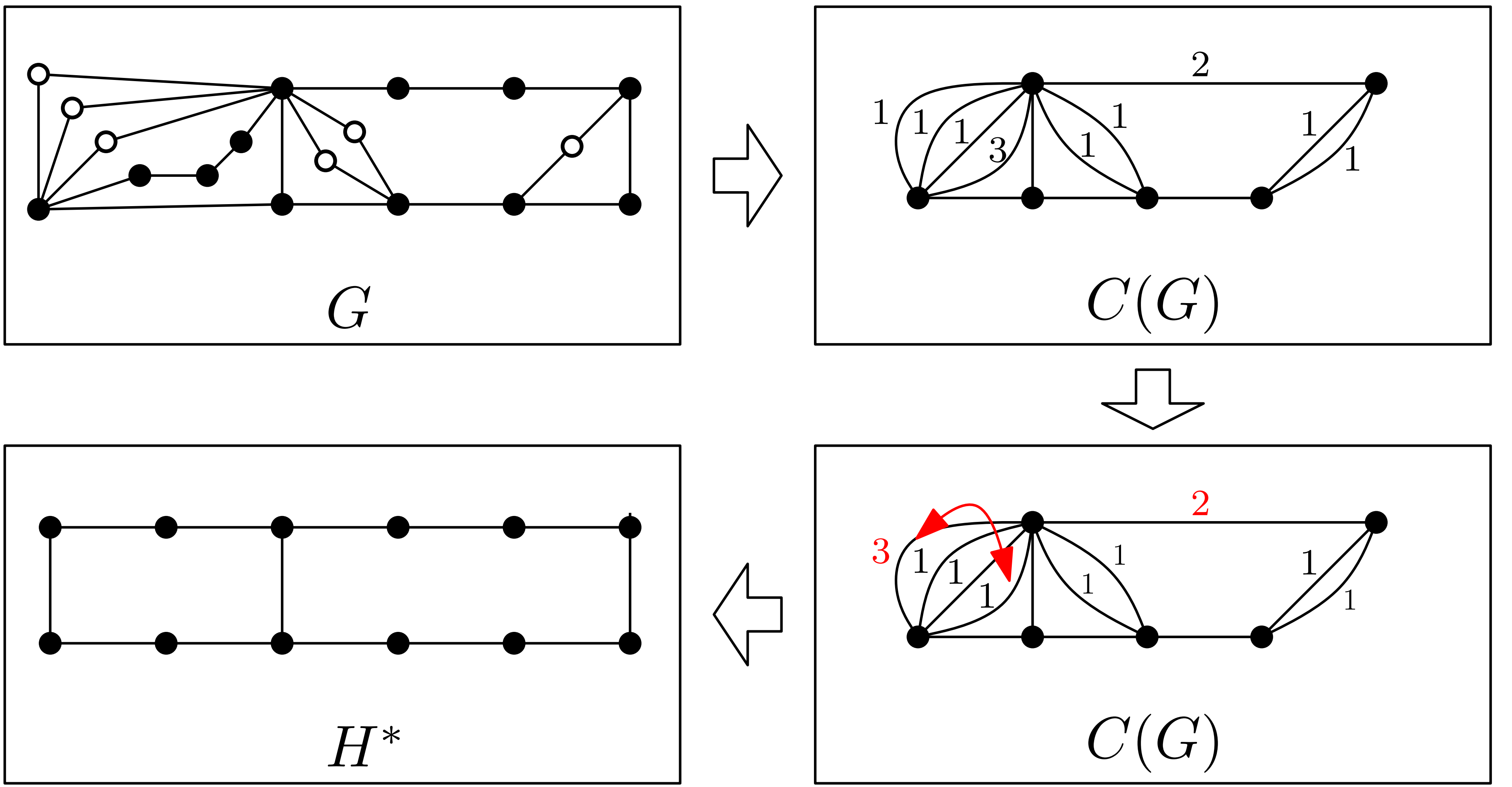
Step 2: Check that all edges with weight > 1 can be embedded on the outer face.

Observation: If so, we found the outer edges of the ladder.



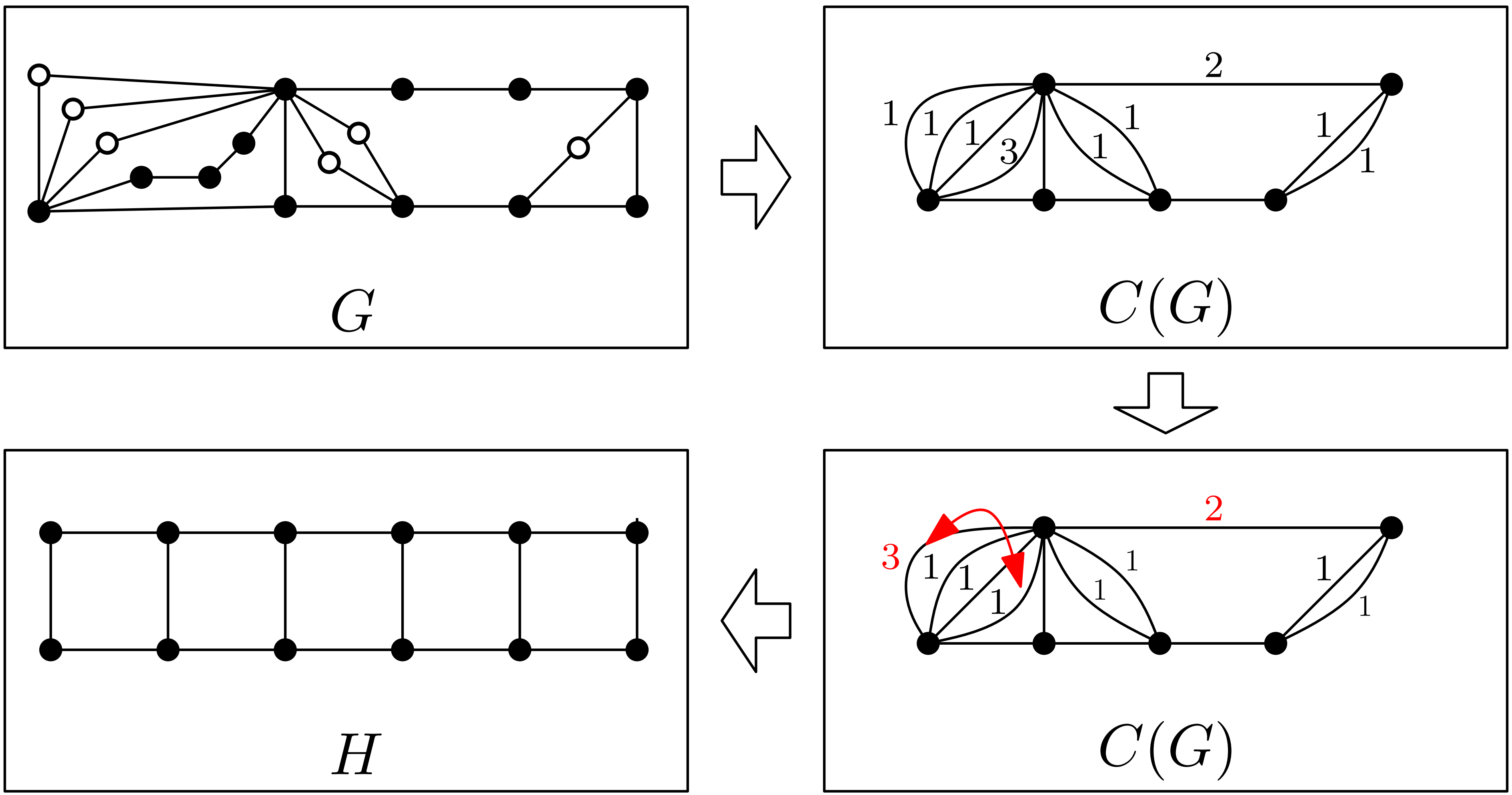
Test for biconnected graphs: Algorithm

Step 3(a): Remove inner edges of weight 1, re-expand outer edges.



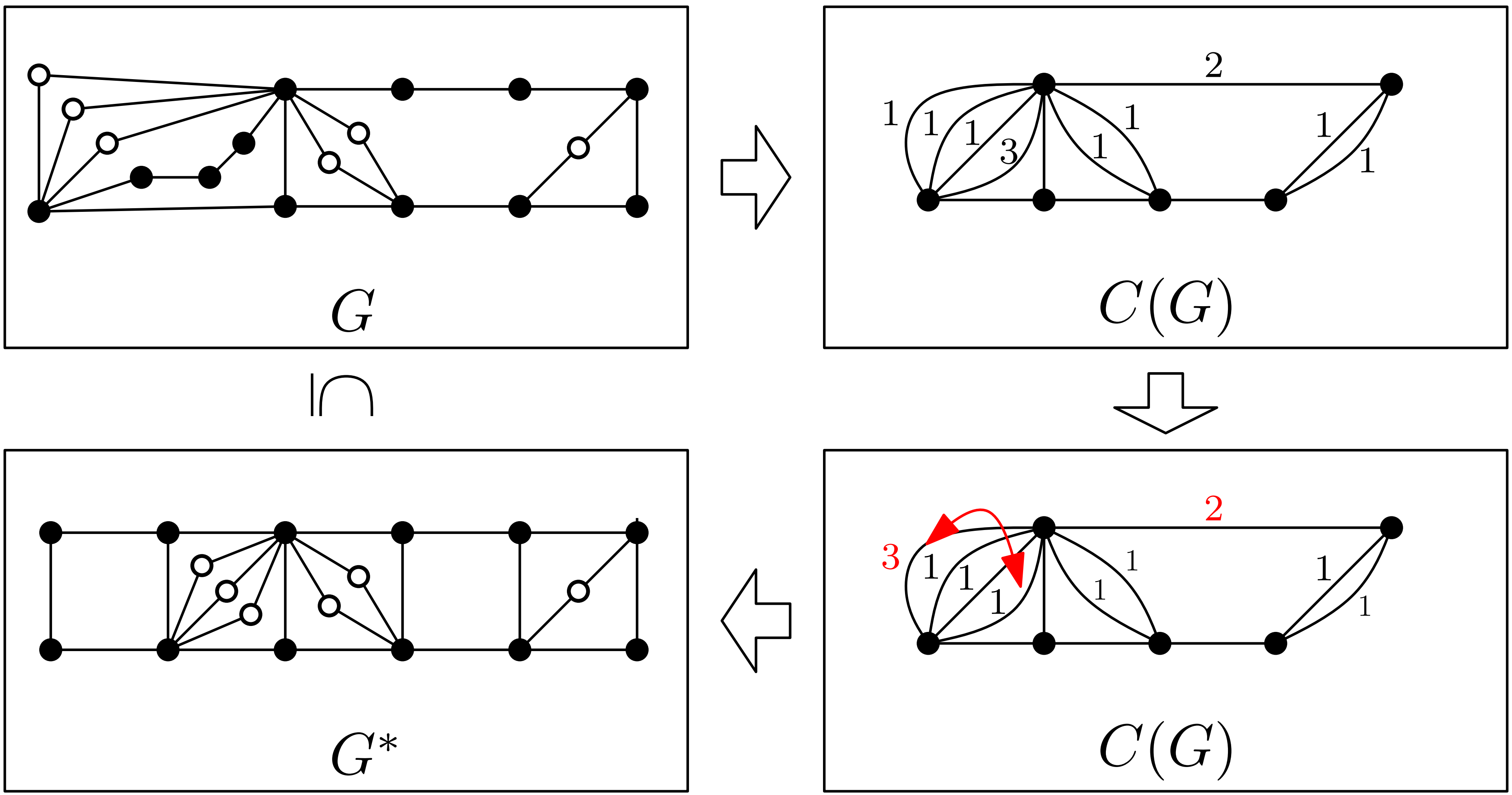
Test for biconnected graphs: Algorithm

Step 3(b): Check if the graph can be augmented to a ladder (Di Giacomo *et al.*, 2014).



Test for biconnected graphs: Algorithm

Step 3(c): Check if the inner paths can be reinserted.



Characterization of 2-layer fan-planar graphs

Stegosaurus: Definition

Definition 2. A *stegosaurus* is recursively defined as follows:

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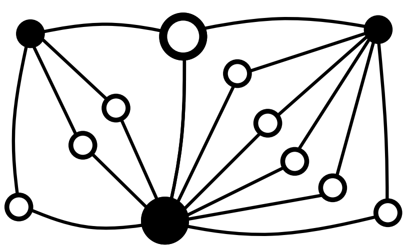
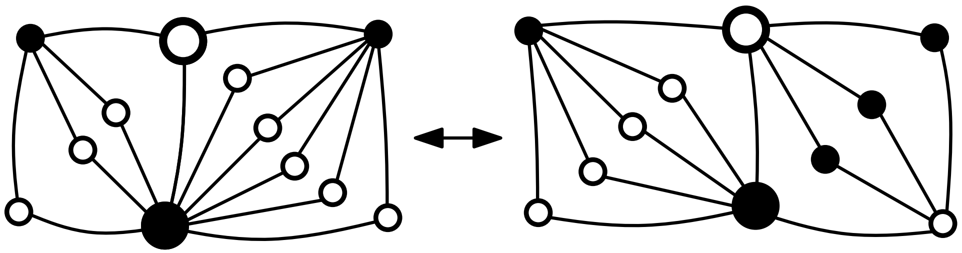


Stegosaurus: Definition

Definition 2. A *stegosaurus* is recursively defined as follows:

- A snake is a stegosaurus;
- The merger of two stegosaurs G_1 and G_2 with respect to vertices v_1 of G_1 and v_2 of G_2 is a stegosaurus.
  A vertex can be merged just once!



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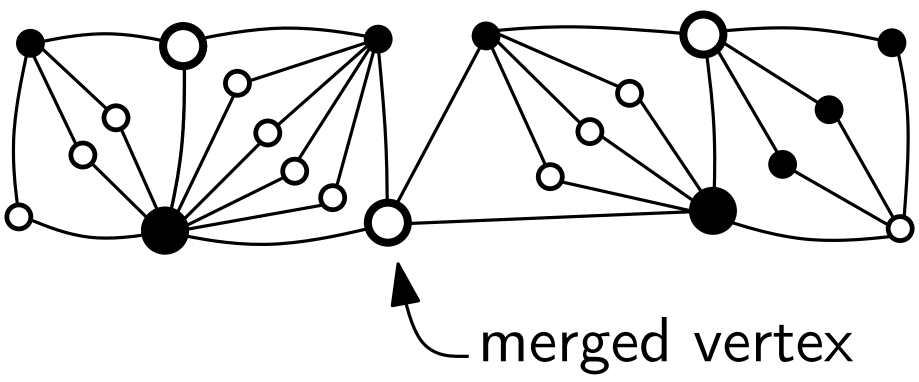

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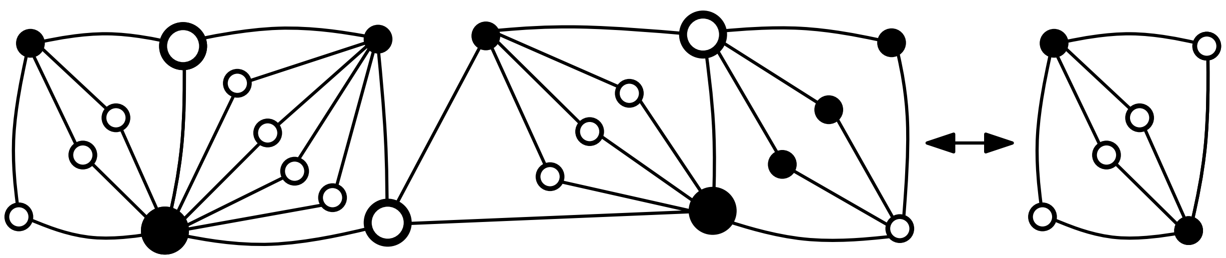

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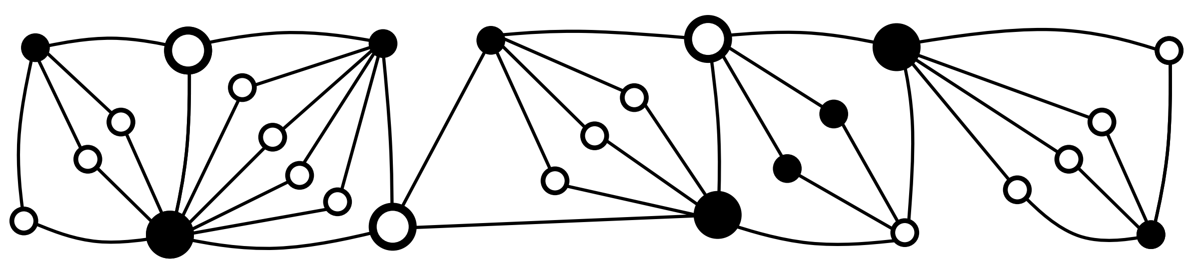

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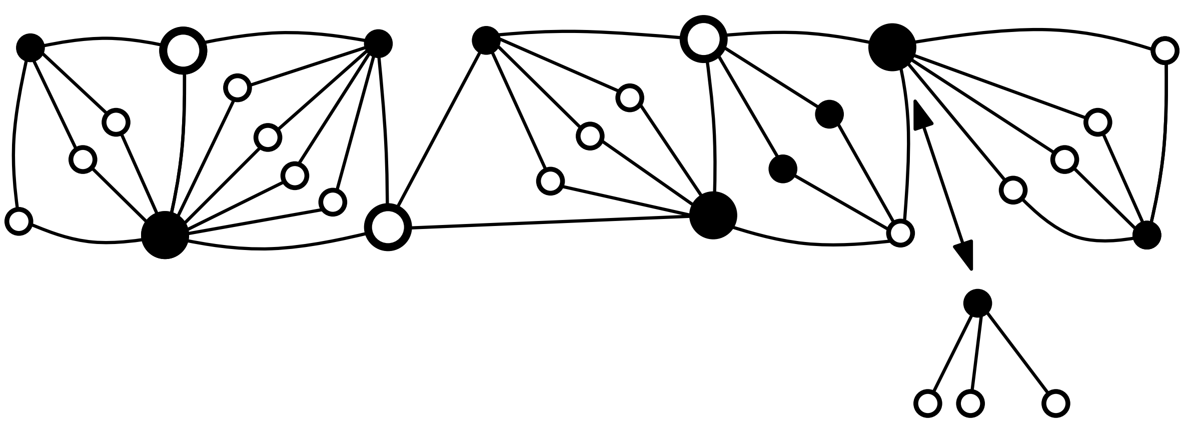

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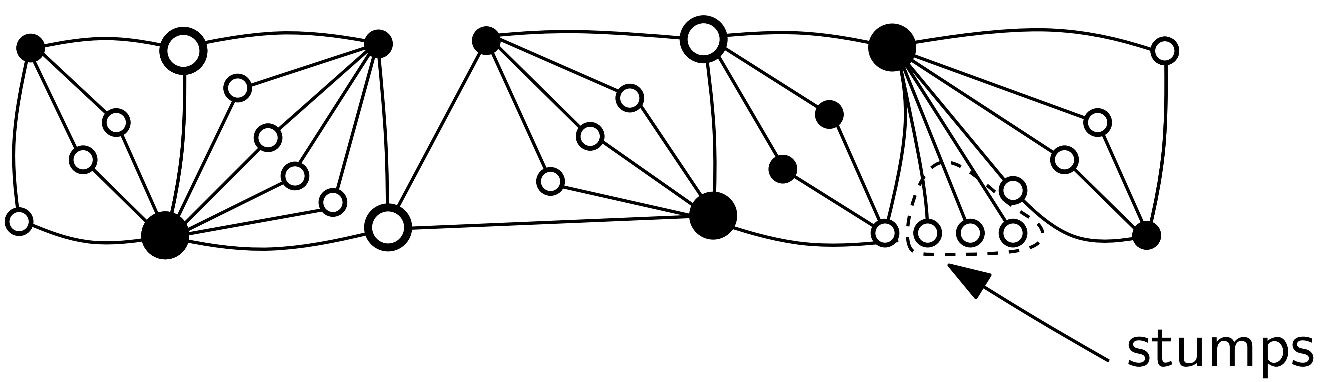
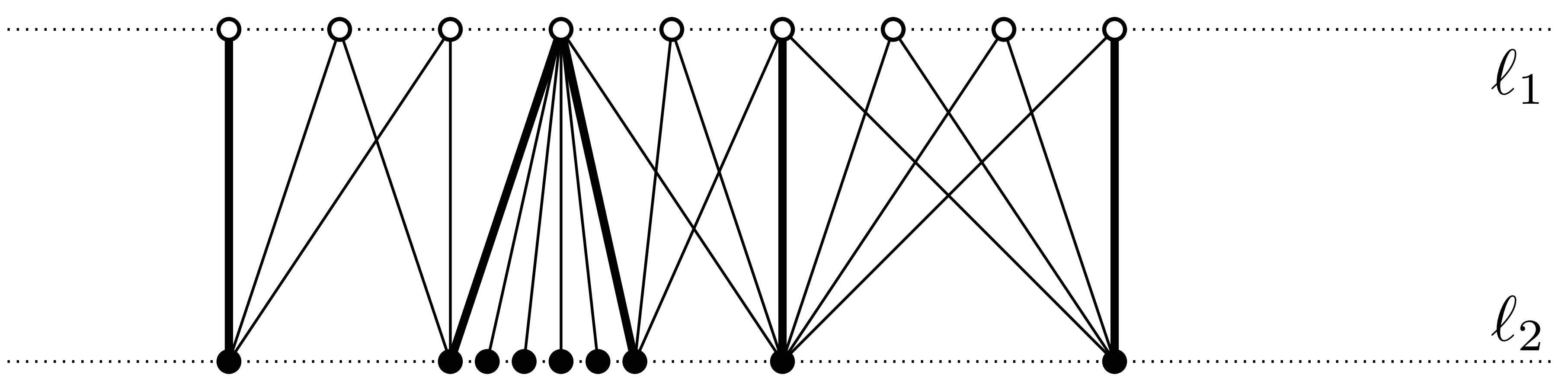


2-Layer Fan-Planar $\leftarrow$ Stegosaurus

Lemma 3 *Every stegosaurus has a 2-layer fan-planar embedding.*

Idea:

- Draw each snake independently
- Merge the drawings
- Draw the stumps



2-Layer Fan-Planar $\rightarrow$ Stegosaurus

Lemma 4 *Let* B *be a block of a* 2*-layer fan-planar graph* G*, and* e *an independent edge, i.e., none of its end-vertices belongs to* B*. No edge of* B *can be crossed by* e *in any* 2*-layer fan-planar embedding of* G*.*

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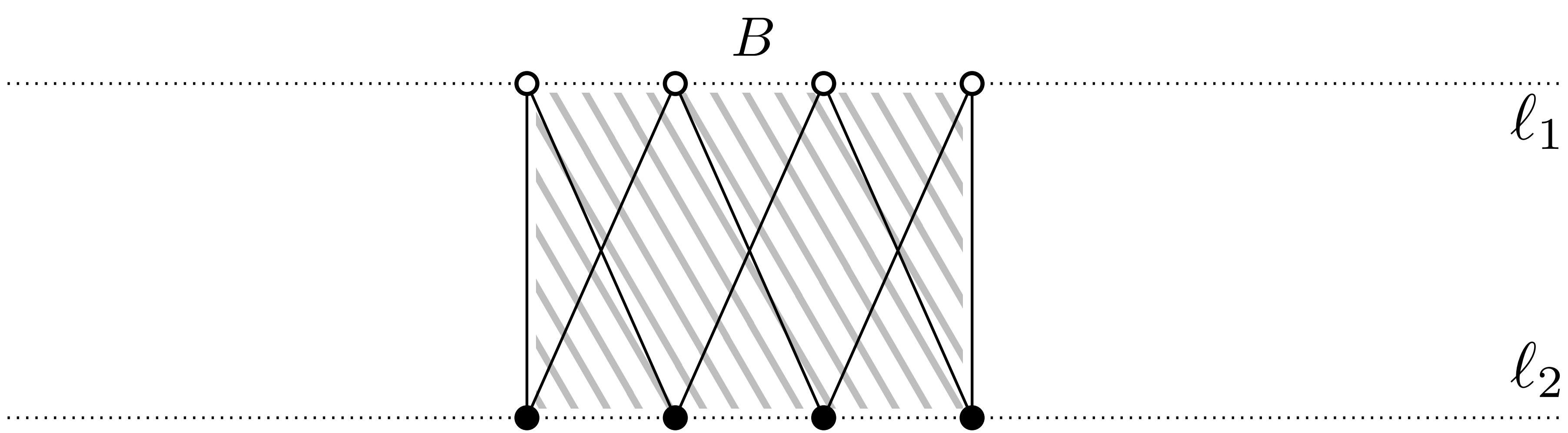
B contains a cycle which has a unique drawing.



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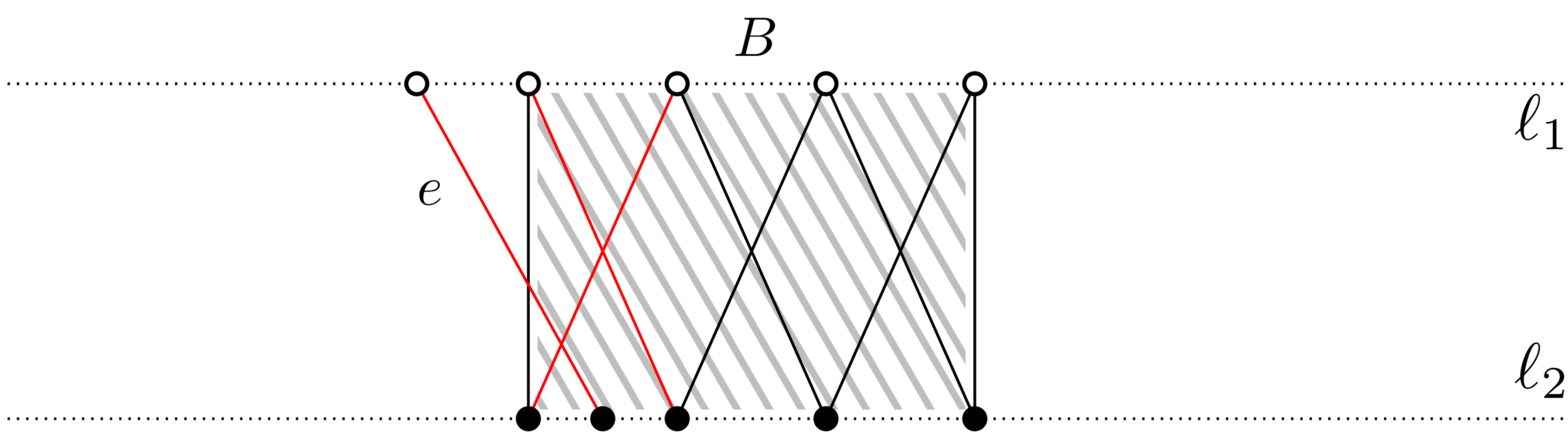

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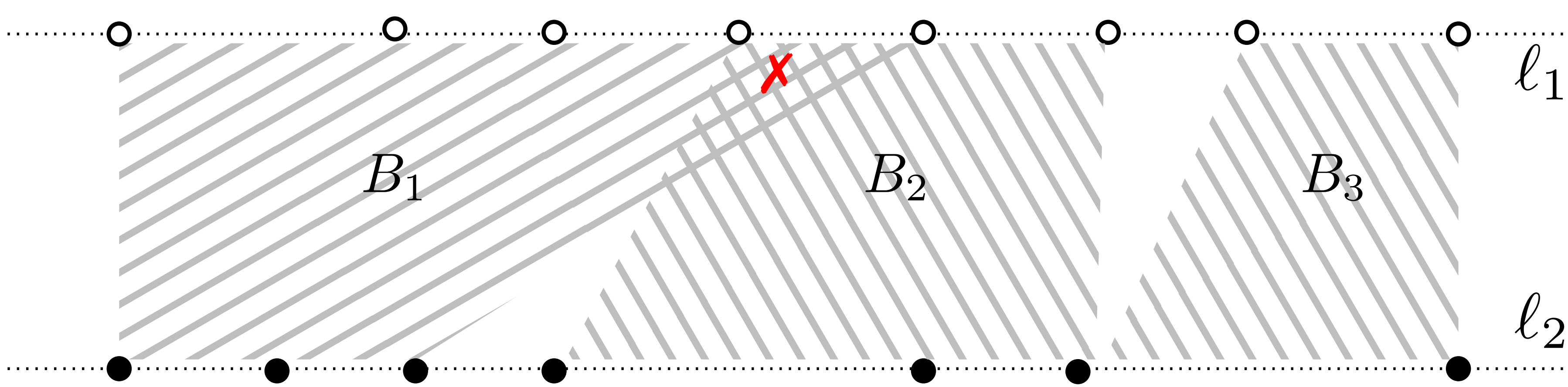
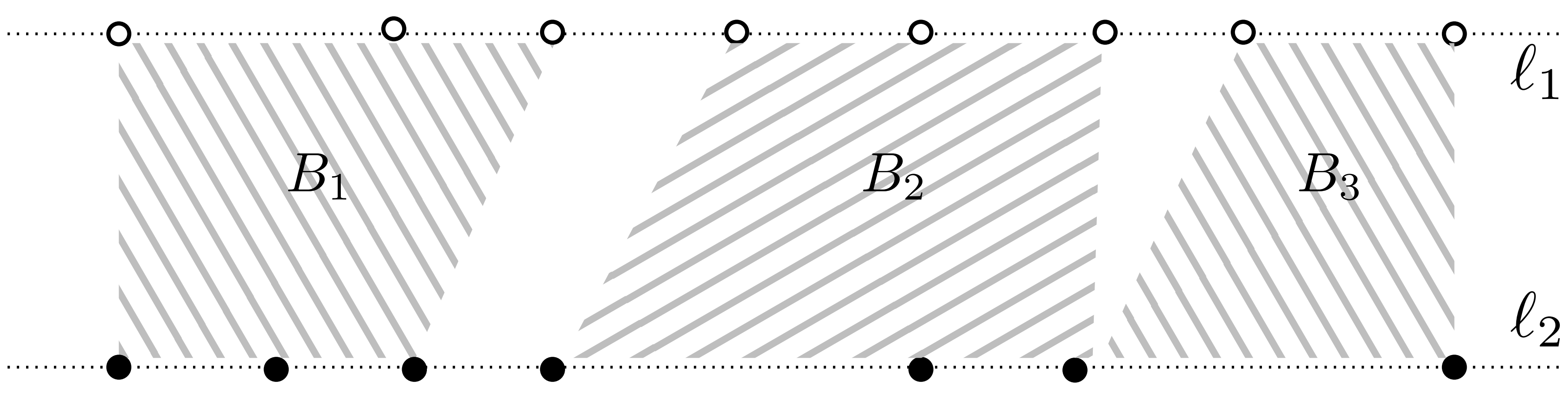
Corollary 1 *In a 2-layer fan-planar embedding, two blocks cannot cross.*



2-Layer Fan-Planar → Stegosaurus

Blocks are "nicely" drawn (**Corollary 1**).



2-Layer Fan-Planar $\rightarrow$ Stegosaurus

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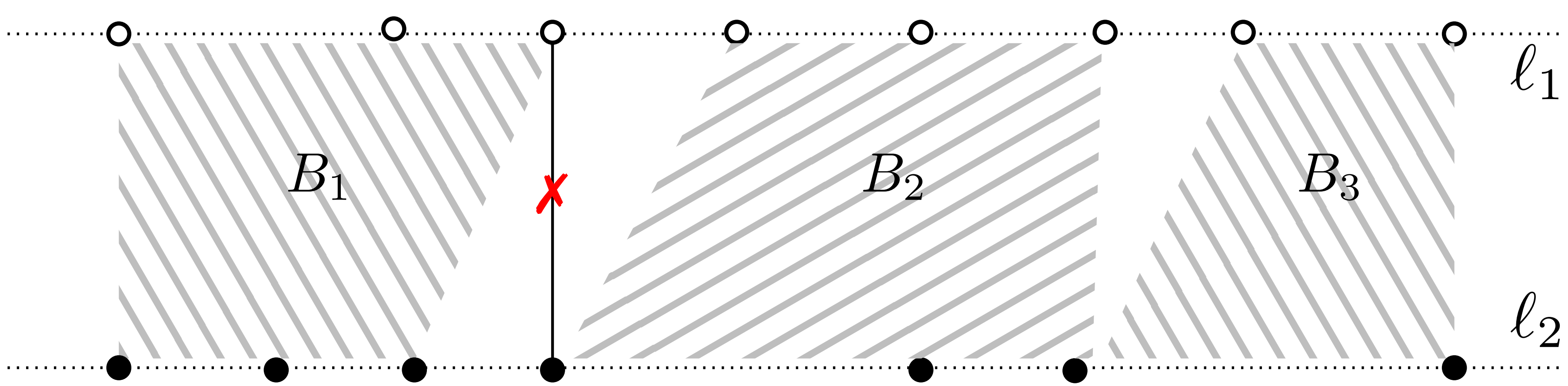
One can show that if G is maximal, then there are no bridges.



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One can show that if G is maximal, then there are no bridges.

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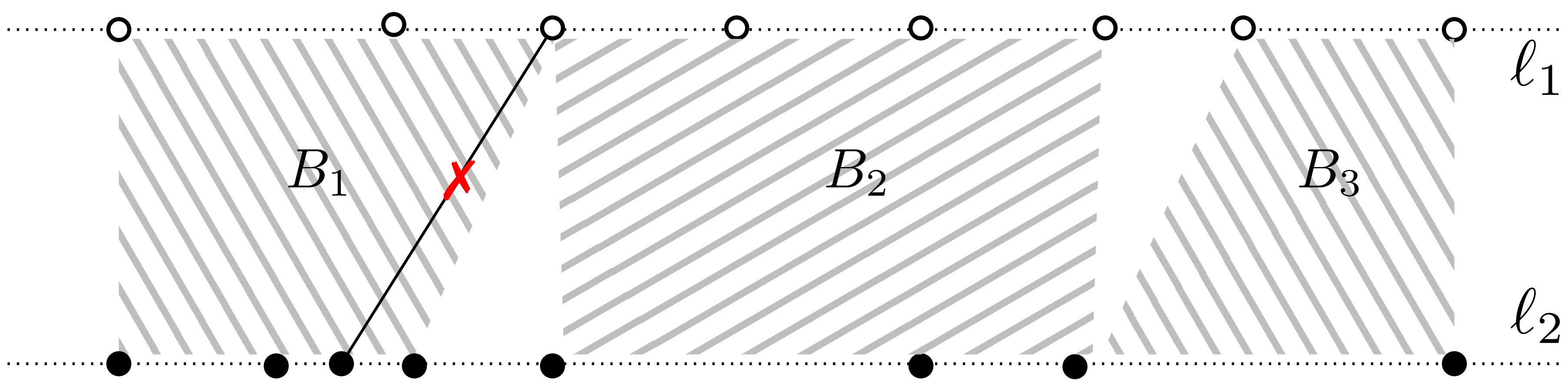

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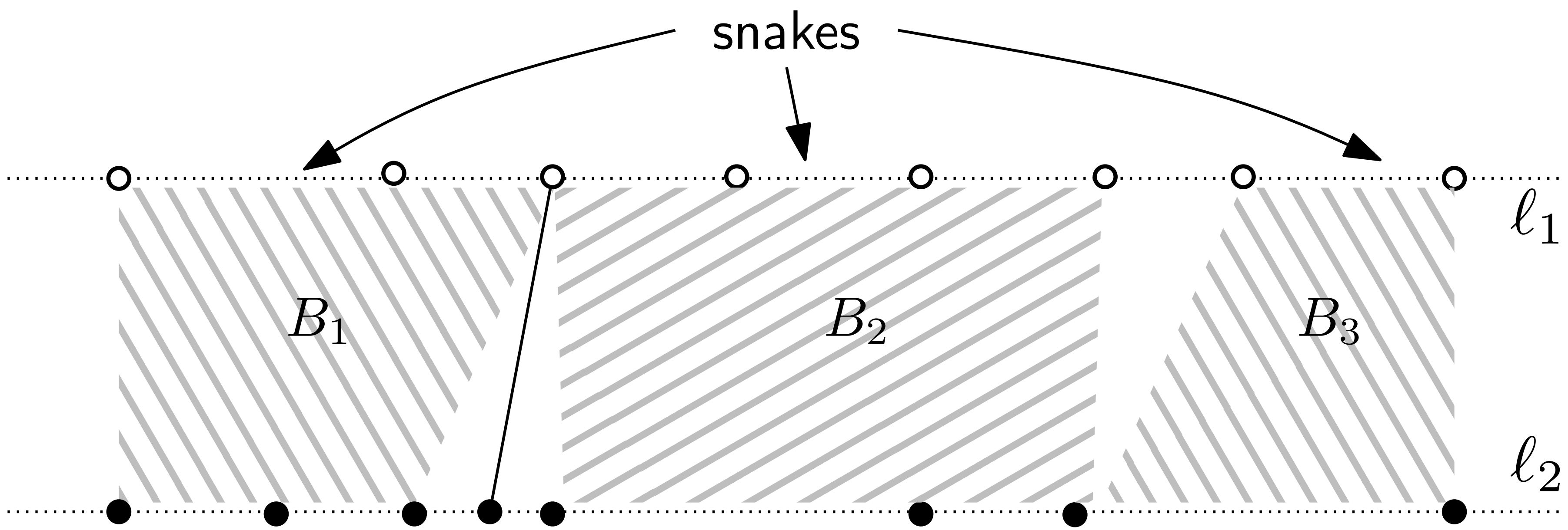

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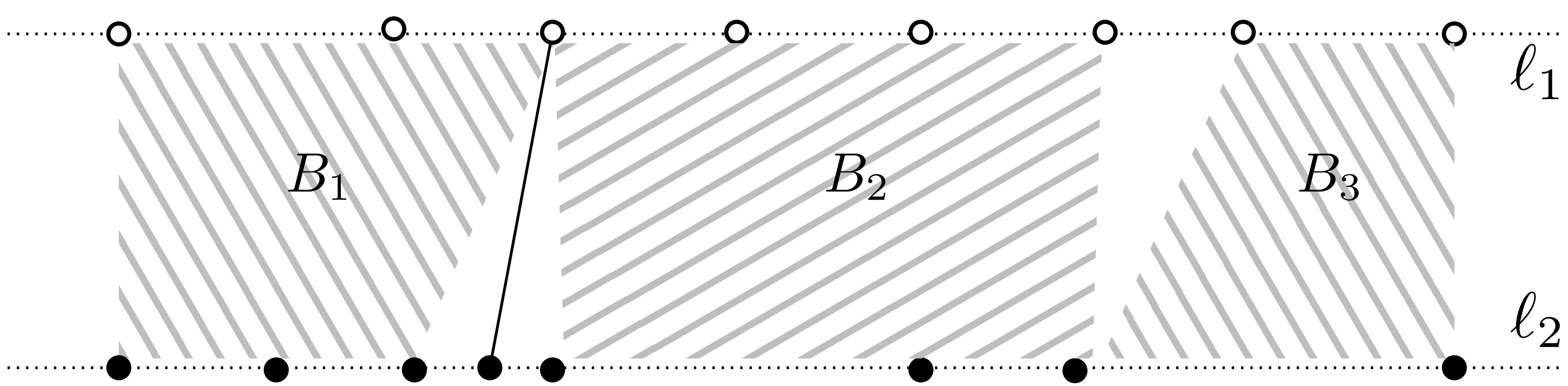
Lemma 5 *Every maximal 2-layer fan-planar graph is a stegosaurus.*



2-Layer Fan-Planar $\iff$ Stegosaurus

Theorem 3 *A graph is 2-layer fan-planar if and only if it is a subgraph of a stegosaurus.*

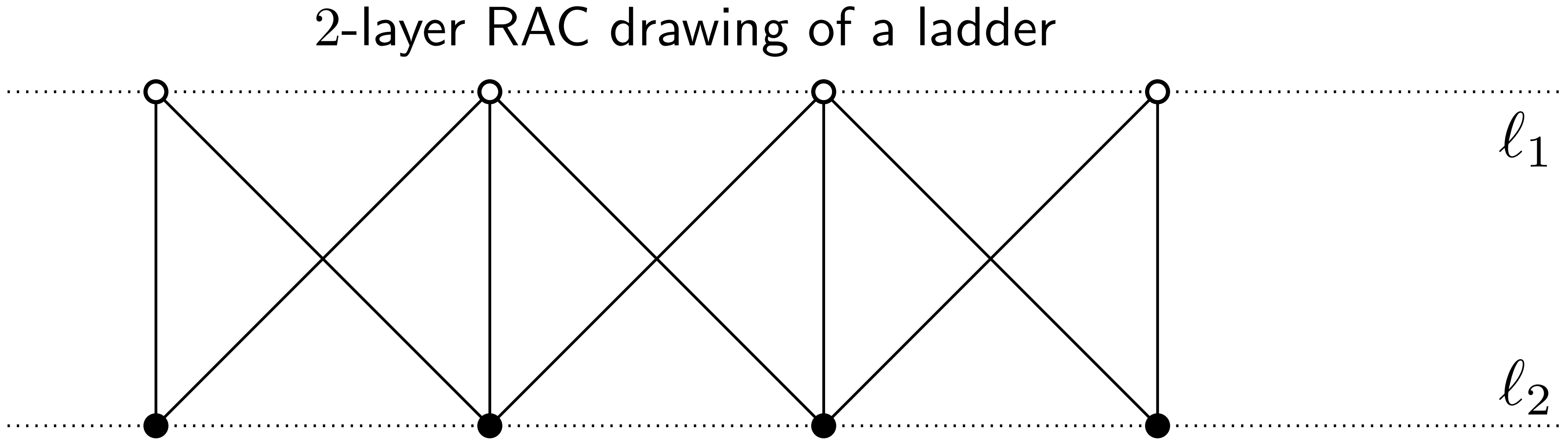
Lemma 3 $+$ Lemma 5.

Relationship with 2-layer RAC graphs

Biconnected Graphs

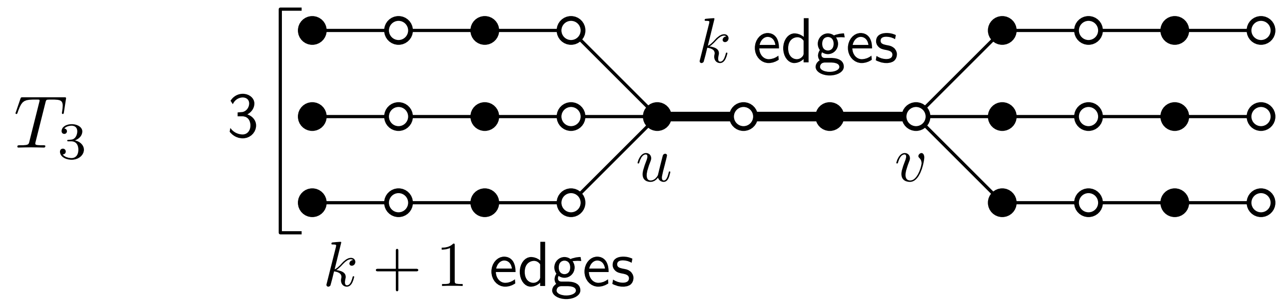
A biconnected graph has a 2-layer RAC embedding if and only if it is a subgraph of a ladder, which is a subgraph of a snake (Di Giacomo *et al.*, 2014).

Corollary 2 *The biconnected 2-layer RAC graphs are a proper subclass of the biconnected 2-layer fan-planar graphs.*



General Graphs

There exist infinitely many trees T_k $(k \geq 3)$ that are 2-layer RAC but not 2-layer fan-planar.



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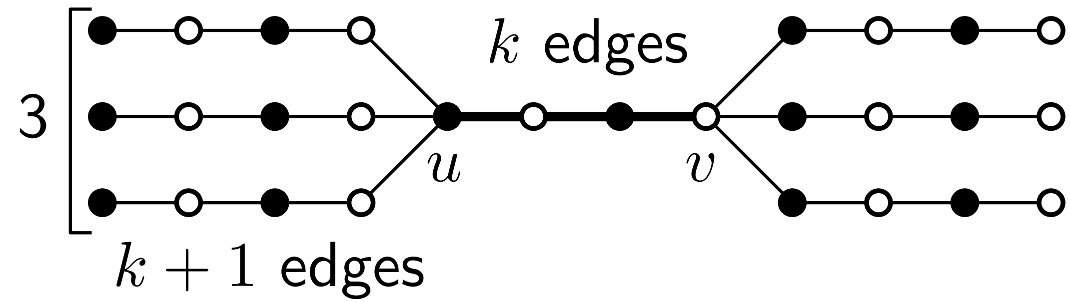
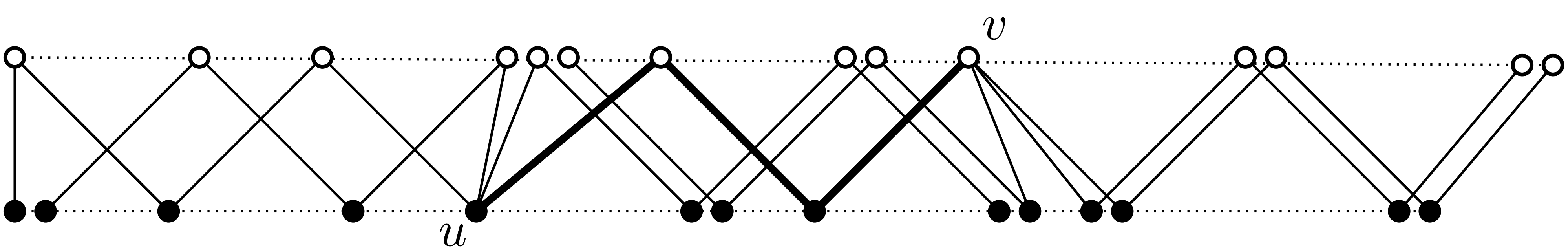
T_3



T_k has a 2-layer RAC embedding.



T_k is not a subgraph of a stegosaurus.

Open Problems

Future Work: How to Attack a Stegosaurus

Test for general graphs

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Test for general graphs

Heuristics for forbidden configurations minimization

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Thank you!