19 POLYTOPE SKELETONS AND PATHS

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INTRODUCTION

The k-dimensional ***skeleton*** of a d-polytope P is the set of all faces of the polytope of dimension at most k. The 1-skeleton of P is called the ***graph*** of P and denoted by $G(P)$. $G(P)$ can be regarded as an abstract graph whose vertices are the vertices of P, with two vertices adjacent if they form the endpoints of an edge of P.

In this chapter, we will describe results and problems concerning graphs and skeletons of polytopes. In Section 19.1 we briefly describe the situation for 3-polytopes. In Section 19.2 we consider general properties of polytopal graphs—subgraphs and induced subgraphs, connectivity and separation, expansion, and other properties. In Section 19.3 we discuss problems related to diameters of polytopal graphs in connection with the simplex algorithm and the Hirsch conjecture. The short Section 19.4 is devoted to polytopal digraphs. Section 19.5 is devoted to skeletons of polytopes, connectivity, collapsibility and shellability, empty faces and polytopes with "few vertices," and the reconstruction of polytopes from their low-dimensional skeletons; we also consider what can be said about the collections of all k-faces of a d-polytope, first for $k = d-1$ and then when k is fixed and d is large compared to k. Section 19.6 describes some results on counting high dimensional polytopes and spheres.

19.1 THREE-DIMENSIONAL POLYTOPES

GLOSSARY

Convex polytopes and their *faces* (and, in particular their *vertices*, *edges*, and *facets*) are defined in Chapter 15 of this Handbook.

A graph is ***d-polytopal*** if it is the graph of some d-polytope.

The following standard graph-theoretic concepts are used: *subgraphs*, *induced subgraphs*, the *complete graph* K_n on n vertices, *cycles*, *trees*, a *spanning tree* of a graph, *degree* of a vertex in a graph, *planar* graphs, *d-connected* graphs, *coloring* of a graph, *subdivision* of a graph, and *Hamiltonian* graphs.

We briefly discuss results on 3-polytopes. Some of the following theorems are the starting points of much research, sometimes of an entire theory. Only in a few cases are there high-dimensional analogues, and this remains an interesting goal for further research.

THEOREM 19.1.1 *Whitney* [Whi32]

Let G be the graph of a 3-polytope P. Then the graphs of faces of P are precisely the induced cycles in G that do not separate G.

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THEOREM 19.1.2 *Steinitz* [Ste22]

A graph G is a graph of a 3-polytope if and only if G is planar and 3-connected.

Steinitz's theorem is the first of several theorems that describe the tame behavior of 3-polytopes. These theorems fail already in dimension four; see Chapter 15.

The theory of planar graphs is a wide and rich theory. Let us quote here the fundamental theorem of Kuratowski.

THEOREM 19.1.3 *Kuratowski* [Kur22, Tho81]

A graph G is planar if and only if G does not contain a subdivision of K_5 or $K_{3,3}$.

THEOREM 19.1.4 *Lipton and Tarjan* [LT79]*, strengthened by Miller* [Mil86]

The graph of every 3-polytope with n vertices can be separated by at most $2\sqrt{2n}$ vertices, forming a circuit in the graph, into connected components of size at most $2n/3$.

It is worth mentioning that the Koebe circle packing theorem gives a new approach to both the Steinitz and Lipton-Tarjan theorems; see [Zie95, PA95].

Euler's formula $V - E + F = 2$ has many applications concerning graphs of 3-polytopes; in higher dimensions, our knowledge of face numbers of polytopes (see Chapter 17) applies to the study of their graphs and skeletons. Simple applications of Euler's theorem are:

THEOREM 19.1.5

Every 3-polytopal graph has a vertex of degree at most 5. (Equivalently, every 3-polytope has a face with at most five sides.)

Zonotopes are centrally symmetric and so are all their faces and therefore, a 3-zonotope must have a face with four sides. This fact is equivalent to the Gallai-Sylvester theorem (Chapter 1).

THEOREM 19.1.6

Every 3-polytope has either a vertex of degree 3 or a triangular face.

A deeper application of Euler's theorem is:

THEOREM 19.1.7 *Kotzig* [Kot55]

Every 3-polytope has two adjacent vertices the sum of whose degrees is at most 13.

For a simple 3-polytope P, let $p_k = p_k(P)$ be the number of k-sized faces of P.

THEOREM 19.1.8 *Eberhard* [Ebe91]

For every finite sequence (p_k) of nonnegative integers with $\sum_{k\geq 3}(6-k)p_k = 12$, there exists a simple 3-polytope P with $p_k(P) = p_k$ for every $k \neq 6$.

Eberhard's theorem is the starting point of a large number of results and problems, see, e.g., [Juc76, Jen93, GZ74]. While no high-dimensional direct analogues are known or even conjectured, the results and problems on facet-forming polytopes and nonfacets mentioned below seem related.

THEOREM 19.1.9 *Motzkin* [Mot64]

The graph of a simple 3-polytope whose facets have $0 \pmod 3$ vertices has, all together, an even number of edges.

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THEOREM 19.1.10 *Barnette* [Bar66]

Every 3*-polytopal graph contains a spanning tree of maximal degree* 3.

We will now describe some results and a conjecture on colorability and Hamiltonian circuits.

THEOREM 19.1.11 *Four Color Theorem: Appel-Haken* [AH76, AH89, RSST97]

The graph of every 3*-polytope is* 4*-colorable.*

THEOREM 19.1.12 *Tutte* [Tut56]

4*-connected planar graphs are Hamiltonian.*

Tait conjectured in 1880, and Tutte disproved in 1946, that the graph of every simple 3-polytope is Hamiltonian. This started a rich theory of cubic planar graphs without large paths.

CONJECTURE 19.1.13 *Barnette*

Every graph of a simple 3*-polytope whose facets have an even number of vertices is Hamiltonian.*

Finally, there are several exact and asymptotic formulas for the numbers of distinct graphs of 3-polytopes. A remarkable enumeration theory was developed by Tutte and was further developed by several authors. We will quote one result.

THEOREM 19.1.14 *Tutte* [Tut62]

The number of rooted simplicial 3*-polytopes with* v *vertices is*

$$\frac{2(4v-11)!}{(3v-7)!(v-2)!}.$$

Tutte's theory also provides efficient algorithms to generate random planar graphs of various types.

PROBLEM 19.1.15

What does a random 3*-polytopal graph look like?*

Motivation to study this problem (and high-dimensional extensions) comes also from physics (specifically, "quantum gravity"). See [ADJ97, Ang02, CS02, GN06, Noy14]. One surprising property of random planar maps of various kinds is that the expected number of vertices of distance at most r from a given vertex behaves like r^4. (Compared to r^2 for the planar grid.)

19.2 GRAPHS OF d-POLYTOPES—GENERALITIES

GLOSSARY

For a graph G, TG denotes any ***subdivision*** of G, i.e., any graph obtained from G by replacing the edges of G by paths with disjoint interiors.

A d-polytope P is ***simplicial*** if all its proper faces are simplices. P is ***simple*** if every vertex belongs to d edges or, equivalently, if the polar of P is simplicial. P is ***cubical*** if all its proper faces are cubes.

A simplicial polytope P is ***stacked*** if it is obtained by the repeated operation of gluing a simplex along a facet.

For the definition of the *cyclic polytope* $C(d, n)$, see Chapter 15.

For two graphs G and H (considered as having disjoint sets V and V' of vertices), $G + H$ denotes the graph on $V \cup V'$ that contains all edges of G and H together with all edges of the form $\{v, v'\}$ for $v \in V$ and $v' \in V'$.

A graph G is ***d-connected*** if G remains connected after the deletion of any set of at most $d - 1$ vertices.

An ***empty simplex*** of a polytope P is a set S of vertices such that S does not form a face but every proper subset of S forms a face.

A graph G whose vertices are embedded in $\mathbb{R}^d$ is ***rigid*** if every small perturbation of the vertices of G that does not change the distance of adjacent vertices in G is induced by an affine rigid motion of $\mathbb{R}^d$. G is ***generically d-rigid*** if it is rigid with respect to "almost all" embeddings of its vertices into $\mathbb{R}^d$. (Generic rigidity is thus a graph theoretic property, but no description of it in pure combinatorial terms is known for $d > 2$; cf. Chapter 61.)

A set A of vertices of a graph G is ***totally separated*** by a set B of vertices, if A and B are disjoint and every path between two distinct vertices in A meets B.

A graph G is an ***ϵ-expander*** if, for every set A of at most half the vertices of G, there are at least $\epsilon \cdot |A|$ vertices not in A that are adjacent to vertices in A.

Neighborly polytopes and $(0, 1)$-*polytopes* are defined in Chapter 15.

The *polar dual* P^{Δ} of a polytope P is defined in Chapter 15.

SUBGRAPHS AND INDUCED SUBGRAPHS

THEOREM 19.2.1 *Grünbaum* [Grü65]

Every d-polytopal graph contains a TK_{d+1}.

For various extensions of this result see [GLM81].

THEOREM 19.2.2 *Kalai* [Kal87]

The graph of a simplicial d-polytope P contains a TK_{d+2} if and only if P is not stacked.

One important difference between the situation for $d = 3$ and for $d > 3$ is that K_n, for every $n > 4$, is the graph of a 4-dimensional polytope (e.g., a cyclic polytope). Simple manipulations on the cyclic 4-polytope with n vertices show:

PROPOSITION 19.2.3 *Perles* (unpublished)

(i) *Every graph G is a spanning subgraph of the graph of a 4-polytope.*

(ii) *For every graph G, $G + K_n$ is a d-polytopal graph for some n and some d.*

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(Here the graph join $G + H$ of two graphs G and H is obtained from putting G and H on disjoint sets of vertices and adding all edges between them.) This proposition extends easily to higher-dimensional skeletons in place of graphs. It is not known what the minimal dimension is for which $G+K_n$ is d-polytopal, nor even whether $G + K_n$ (for some $n = n(G)$) can be realized in some bounded dimension uniformly for all graphs G.

CONNECTIVITY AND SEPARATION

THEOREM 19.2.4 *Balinski* [Bal61]

The graph of a d-polytope is d-connected.

A set S of d vertices that separates P must form an empty simplex; in this case, P can be obtained by gluing two polytopes along a simplex facet of each.

A graph G is k-linked if for every two disjoint sequences $(v_1, v_2, \ldots, v_k)$ and $(w_1, w_2, \ldots w_k)$ of vertices of G, there are e vertex-disjoint paths connecting v_i to w_i, $i = 1, 2, \ldots, k$.

THEOREM 19.2.5 *Larman and Mani* [LM70]

A graph of a d-polytope is $\lfloor (d+1)/3 \rfloor$-linked

Let $k(d)$ be the smallest integer to that every graph of a d-polytopes is $k(d)$-linked. Larman conjectured that $k(d) \geq \lfloor d/2 \rfloor$. However, Gallivan [Gal85]. showed that $k(d) \leq [(2d + 3)/5]$.

PROBLEM 19.2.6 *Understand the behavior of $k(d)$.*

THEOREM 19.2.7 *Cauchy, Dehn, Aleksandrov, Whiteley, ...*

(i) *Cauchy's theorem: If P is a simplicial d-polytope, $d \geq 3$, then $G(P)$ (with its embedding in $\mathbb{R}^d$) is rigid.*

(ii) *Whiteley's theorem* [Whi84]*: For a general d-polytope P, let G' be a graph (embedded in $\mathbb{R}^d$) obtained from $G(P)$ by triangulating the 2-faces of P without introducing new vertices. Then G' is rigid.*

COROLLARY 19.2.8

For a simplicial d-polytope P, $G(P)$ is generically d-rigid. For a general d-polytope P and a graph G' (considered as an abstract graph) as in the previous theorem, G' is generically d-rigid.

The main combinatorial application of the above theorem is the Lower Bound Theorem (see Chapter 17) and its extension to general polytopes.

Note that Corollary 19.2.8 can be regarded also as a strong form of Balinski's theorem. It is well known and easy to prove that a generic d-rigid graph is d-connected. Therefore, for simplicial (or even 2-simplicial) polytopes, Corollary 19.2.8 implies directly that $G(P)$ is d-connected.

For general polytopes we can derive Balinski's theorem as follows. Suppose to the contrary that the graph G of a general d-polytope P is not d-connected and therefore its vertices can be separated into two parts (say, red vertices and blue vertices) by deleting a set A of $d-1$ vertices. It is easy to see that every 2-face of P

can be triangulated without introducing a blue-red edge. Therefore, the resulting triangulation is not $(d-1)$-connected and hence it is not generically d-rigid. This contradicts the assertion of Corollary 19.2.8.

Let $\mu(n, d) = f_{d-1}(C(d, n))$ be the number of facets of a cyclic d-polytope with n vertices, which, by the Upper Bound Theorem, is the maximal number of facets possible for a d-polytope with n vertices.

THEOREM 19.2.9 *Klee* [Kle64]

The number of vertices of a d-polytope that can be totally separated by n vertices is at most $\mu(n, d)$.

Klee also showed by considering cyclic polytopes with simplices stacked to each of their facets that this bound is sharp. It follows that there are graphs of simplicial d-polytopes that are not graphs of $(d-1)$-polytopes. (After realizing that the complete graphs are 4-polytopal one's naive thought might be that every d-polytopal graph is 4-polytopal.)

EXPANSION

Expansion properties for the graph of the d-dimensional cube are known and important in various areas of combinatorics. By direct combinatorial methods, one can obtain expansion properties of duals to cyclic polytopes. There are a few positive results and several interesting conjectures on expansion properties of graphs of large families of polytopes.

THEOREM 19.2.10 *Kalai* [Kal91]

Graphs of duals to neighborly d-polytopes with n facets are ϵ-expanders for $\epsilon = O(n^{-4})$.

This result implies that the diameter of graphs of duals to neighborly d-polytopes with n facets is $O(d \cdot n^4 \cdot \log n)$.

CONJECTURE 19.2.11 *Mihail and Vazirani* [FM92, Kai01]

Graphs of $(0, 1)$-polytopes P have the following expansion property: For every set A of at most half the vertices of P, the number of edges joining vertices in A to vertices not in A is at least $|A|$.

Dual graphs to cyclic $2k$-polytopes with n vertices for n large look somewhat like graphs of grids in $\mathbb{Z}^k$ and, in particular, have no separators of size $o(n^{1-1/k})$. It was conjectured that graphs of polytopes cannot have very good expansion properties, namely that the graph of every simple d-polytope with n vertices can be separated into two parts, each having at least $n/3$ vertices, by removing $O(n^{1-1/(d-1)})$ vertices. However,

THEOREM 19.2.12 *Loiskekoski and Ziegler* [LZ15]

There are simple 4-dimensional polytopes with n vertices such that all separators of the graph have size at least $\Omega(n/\log^{3/2} n)$.

It is still an open problem if there are examples of simple d-polytopes with n vertices $n \to \infty$ whose graphs have no separators of size $o(n)$, or even of simple d-polytopes with n vertices, whose graphs are expanders.

OTHER PROPERTIES

CONJECTURE 19.2.13 *Barnette*

Every graph of a simple d-polytope, $d \geq 4$, is Hamiltonian.

THEOREM 19.2.14

For a simple d-polytope P, $G(P)$ is 2-colorable if and only if $G(P^{\Delta})$ is d-colorable.

This theorem was proved in an equivalent form for $d = 4$ by Goodman and Onishi [GO78]. (For $d = 3$ it is a classical theorem by Ore.) For the general case, see Joswig [Jos02]. This theorem is related to seeking two-dimensional analogues of Hamiltonian cycles in skeletons of polytopes and manifolds; see [Sch94].

19.3 DIAMETERS OF POLYTOPAL GRAPHS

GLOSSARY

A ***d-polyhedron*** is the intersection of a finite number of halfspaces in $\mathbb{R}^d$.

$\Delta(d, n)$ denotes the maximal diameter of the graphs of d-dimensional polyhedra P with n facets. (Here again vertices correspond to 0-faces and edges correspond to 1-faces.)

$\Delta_{\rm b}(d, n)$ denotes the maximal diameter of the graphs of d-polytopes with n vertices.

Given a d-polyhedron P and a linear functional ϕ on $\mathbb{R}^d$, we denote by $G^{\rightarrow}(P)$ the directed graph obtained from $G(P)$ by directing an edge $\{v, u\}$ from v to u if $\phi(v) \leq \phi(u)$. $v \in P$ is a ***top vertex*** if ϕ attains its maximum value in P on v.

Let $H(d, n)$ be the maximum over all d-polyhedra with n facets and all linear functionals on $\mathbb{R}^d$ of the maximum length of a minimal monotone path from any vertex to a top vertex.

Let $M(d, n)$ be the maximal number of vertices in a monotone path over all d-polyhedra with n facets and all linear functionals on $\mathbb{R}^d$.

For the notions of *simplicial complex*, *polyhedral complex*, *pure simplicial complex*, *flag complex* and the *boundary complex* of a polytope, see Chapter 17.

Given a pure $(d-1)$-dimensional simplicial (or polyhedral) complex K, the ***dual graph*** $G^{\Delta}(K)$ of K is the graph whose vertices are the facets ($(d-1)$-faces) of K, with two facets F, F' adjacent if $\dim(F \cap F') = d - 2$.

A pure simplicial complex K is ***vertex-decomposable*** if there is a vertex v of K such that $\mathrm{lk}(v) = \{S \backslash \{v\} \mid S \in K, v \in S\}$ and $\mathrm{ast}(v) = \{S \mid S \in K, v \notin S\}$ are both vertex-decomposable. (The complex $K = \{\emptyset\}$ consisting of the empty face alone is vertex-decomposable.)

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LOWER BOUNDS

It is a long-outstanding open problem to determine the behavior of the function $\Delta(d,n)$ and $\Delta_b(d,n)$. See [San13b] for a recent surveys on this problem. In 1957, Hirsch conjectured that $\Delta(d,n) \le n-d$. Klee and Walkup [KW67] showed that the Hirsch conjecture is false for unbounded polyhedra.

THEOREM 19.3.1 *Klee and Walkup*

$$\Delta(d,n) \ge n-d+\min\{\lfloor d/4\rfloor, \lfloor (n-d)/4\rfloor\}.$$

The Hirsch conjecture for bounded polyhedra remained open until 2010. The special case asserting that $\Delta_b(d,2d)=d$ is called the ***d-step conjecture***, and it was shown by Klee and Walkup to imply that $\Delta_b(d,n) \le n-d$. Another equivalent formulation is that between any pair of vertices v and w of a polytope P there is a ***nonrevisiting path***, i.e., a path $v=v_1,v_2,\dots,v_m=w$ such that for every facet F of P, if $v_i,v_j \in F$ for $i<j$ then $v_k \in F$ for every k with $i \le k \le j$.

THEOREM 19.3.2 *Holt-Klee* [HK98a, HK98b, HK98c]*, Fritzsche-Holt* [FH99]

For $n > d \ge 8$

$$\Delta_b(d,n) \ge n-d.$$

THEOREM 19.3.3 *Santos* [San13a], Matschke, Santos, and Weibel [MSW15]

For $d \ge 20$

$$\Delta_b(d,2d) > d.$$

Santos' argument uses a beautiful extension of the nonrevisiting conjecture (in its dual form) for certain *nonsimplicial* polytopes. A counterexample for this conjecture is found (already in dimension five) and it leads to a counterexample of the d-step conjecture. Matschke, Santos, and Weibel [MSW15], improved the construction and it now applies for $d \ge 20$.

UPPER BOUNDS

The known upper bounds for $\Delta(d,n)$ when d is fixed and n is large.

THEOREM 19.3.4 *Larman* [Lar70]

$$\Delta(d,n) \le n \cdot 2^{d-3}.$$

And a slight improvement by Barnette

THEOREM 19.3.5 *Barnette* [Bar74]

$$\Delta(d,n) \le \frac{2}{3}\cdot(n-d+5/2)\cdot 2^{d-3}.$$

When n is not very large w.r.t. d the best known upper bounds are quasipolynomials.

THEOREM 19.3.6 *Kalai and Kleitman* [KK92]

$$\Delta(d,n) \le n \cdot \binom{\log n + d}{d} \le n^{\log d + 1}.$$

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Recently, a small improvement was made by Todd [Tod14] who proved that $\Delta(d,n) \le (n-d)^{\log d}$.

The major open problem in this area is:

PROBLEM 19.3.7

Is there a polynomial upper bound for $\Delta(d,n)$? Is there a linear upper bound for $\Delta(d,n)$?

Even upper bounds of the form $\exp c \log d \log(n-d)$ and $n2^{cn}$ for some $c < 1$ would be a major progress.

CONJECTURE 19.3.8 *Hähnle, [Häh10, San13b]*

$$\Delta(d,n) \le d(n-1)$$

Many of the upper bounds for $\Delta(d,n)$ applies to much more general combinatorial objects [Kal92, EHRR10]. Hähnle's conjecture applies to an abstract (much more general) setting of the problem formulated for certain families of monomials, and Hähnle's proposed bound is tight in this greater generality.

SPECIAL CLASSES OF POLYHDRA

Some special classes of polytopes are known to satisfy the Hirsch bound or to have upper bounds for their diameters that are polynomial in d and n.

THEOREM 19.3.9 *Provan and Billera* [PB80]

Let G be the dual graph that corresponds to a vertex-decomposable $(d-1)$-dimensional simplicial complex with n vertices. Then the diameter of G is at most $n-d$.

There are simplicial polytopes whose boundary complexes are not vertex-decomposable. Lockeberg found such an example in 1977 and examples that violate much weaker forms of vertex decomposability were found in [DK12, HPS14].

THEOREM 19.3.10 *Naddef* [Nad89]

The graph of every $(0,1)$ d-polytope has diameter at most d.

Balinski [Bal84] proved the Hirsch bound for dual transportation polytopes, Dyer and Frieze [DF94] showed a polynomial upper bound for unimodular polyhedra; for a recent improved bounds see [BDE$^+$14]. Kalai [Kal92] proved that if the ratio between the number of facets and the dimension is bounded above for the polytope and all its faces then the diameter is bounded above by a polynomial in the dimension, Kleinschmidt and Onn [KO92] proved extensions of Naddef's results to integral polytopes, and Deza, Manoussakis and Onn [DMO17] conjectured a sharp version of their result attained by Minkowski sum of primitive lattice vectors. Deza and Onn [DO95] found upper bounds for the diameter in terms of lattice points in the polytope. A recent result based on the basic fact that locally convex sets of small intrinsic diameter in CAT(1) spaces are convex is:

THEOREM 19.3.11 *Adiprasito and Benedetti* [AB14]

Flag simplicial spheres satisfy the Hirsch conjecture.

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LINEAR PROGRAMMING AND ROUTING

The value of $\Delta(d, n)$ is a lower bound for the number of iterations needed in the worst case for Dantzig's simplex algorithm for linear programming with any pivot rule. However, it is still an open problem to find pivot rules where each pivot step can be computed with a polynomial number of arithmetic operations in d and n such that the number of pivot steps needed comes close to the upper bounds for $\Delta(d, n)$ given above. See Chapter 49.

We note that a continuous analog of the Hirsch conjecture proposed and studied by Deza, Terlaky, and Zinchenko [DTZ09] was disproved by Allamigeon, Benchimol, Gaubert, and Joswig [ABGJ17]. The counterexample is a polytope obtained via tropical geometry.

The problem of routing in graphs of polytopes, i.e., finding a path between two vertices, is an interesting computational problem.

PROBLEM 19.3.12

Find an efficient routing algorithm for convex polytopes.

Using linear programming it is possible to find a path in a polytope P between two vertices that obeys the upper bounds given above such that the number of calls to the linear programming subroutine is roughly the number of edges of the path. Finding a routing algorithm for polytopes with a "small" number of arithmetic operations as a function of d and n is an interesting challenge. The subexponential simplex-type algorithms (see Chapter 49) yield subexponential routing algorithms, but improvement for routing beyond what is known for linear programming may be possible.

The upper bounds for $\Delta(d, n)$ mentioned above apply even to $H(d, n)$. Klee and Minty considered a certain geometric realization of the d-cube to show that

THEOREM 19.3.13 *Klee and Minty* [KM72]

$M(d, 2d) \geq 2^d$.

Far-reaching extensions of the Klee-Minty construction were found by Amenta and Ziegler [AZ99]. It is not known for $d > 3$ and $n \geq d+3$ what the precise upper bound for $M(d, n)$ is and whether it coincides with the maximum number of vertices of a d-polytope with n facets given by the upper bound theorem (Chapter 17).

19.4 POLYTOPAL DIGRAPHS

Given a d-polytope P and a linear objective function ϕ not constant on edges, direct every edge of $G(P)$ towards the vertex with the higher value of the objective function. A directed graph obtained in this way is called a ***polytopal digraph***.

The following basic result is fundamental for the simplex algorithm and also has many applications for the combinatorial theory of polytopes.

THEOREM 19.4.1 *Folklore* (see, e.g., [Wil88])

A polytopal digraph has one sink (and one source). Moreover, every induced subgraph on the vertices of any face of the polytope has one sink (and one source).

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An acyclic orientation of $G(P)$ with the property that every face has a unique sink is called an ***abstract objective function***. Joswig, Kaibel, and Körner [JKK02] showed that an acyclic orientation for which every 2-dimensional face has a unique sink is already an abstract objective function.

The h-vector of a simplicial polytope P has a simple and important interpretation in terms of the directed graph that corresponds to the polar of P. The number $h_k(P)$ is the number of vertices v of P^Δ of outdegree k. (Recall that every vertex in a simple polytope has exactly d neighboring vertices.) Switching from ϕ to $-\phi$, one gets the Dehn-Sommerville relations $h_k = h_{d-k}$ (including the Euler relation for $k = 0$); see Chapter 17.

Studying polytopal digraphs and digraphs obtained by abstract objective functions is very interesting in three- and higher dimensions. Digraphs whose underlying simple graphs are that of the cube are of special interest and important in the theory of linear programming. The work of Friedmann, Hansen, and Zwick [FHZ11] who found very general constructions based on certain stochastic games is of particular importance. (See Chapter 49.) In three dimensions, polytopal digraphs admit a simple characterization:

THEOREM 19.4.2 *Mihalisin and Klee* [MK00]

Suppose that K is an orientation of a 3-polytopal graph G. Then the digraph K is 3-polytopal if and only if it is acyclic, has a unique source and a unique sink, and admits three independent monotone paths from the source to the sink.

Mihalisin and Klee write in their article "we hope that the present article will open the door to a broader study of polytopal digraphs."

19.5 SKELETONS OF POLYTOPES

GLOSSARY

A pure polyhedral complex K is ***strongly connected*** if its dual graph is connected.

A ***shelling order*** of the facets of a polyhedral $(d-1)$-dimensional sphere is an ordering of the set of facets $F_1, F_2, \ldots, F_n$ such that the simplicial complex K_i spanned by $F_1 \cup F_2 \cup \cdots \cup F_i$ is a simplicial ball for every $i < n$. A polyhedral complex is ***shellable*** if there exists a shelling order of its facets.

A simplicial polytope is ***extendably shellable*** if any way to start a shelling can be continued to a shelling.

An ***elementary collapse*** on a simplicial complex is the deletion of two faces F and G so that F is maximal and G is a codimension-1 face of F that is not included in any other maximal face. A polyhedral complex is ***collapsible*** if it can be reduced to the void complex by repeated applications of elementary collapses.

A d-dimensional polytope P is ***facet-forming*** if there is a $(d+1)$-dimensional polytope Q such that all facets of Q are combinatorially isomorphic to P. If no such Q exists, P is called a ***nonfacet***.

A ***rational polytope*** is a polytope whose vertices have rational coordinates. (Not every polytope is combinatorially isomorphic to a rational polytope; see Chapter 15.)

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A d-polytope P is ***k-simplicial*** if all its faces of dimension at most k are simplices. P is ***k-simple*** if its polar dual P^Δ is k-simplicial.

Zonotopes are defined in Chapters 15 and 17.

Let K be a polyhedral complex. An ***empty simplex*** S of K is a minimal nonface of K, i.e., a subset S of the vertices of K with S itself not in K, but every proper subset of S in K.

Let K be a polyhedral complex and let U be a subset of its vertices. The ***induced subcomplex*** of K on U, denoted by $K[U]$, is the set of all faces in K whose vertices belong to U. An ***empty face*** of K is an induced polyhedral subcomplex of K that is homeomorphic to a polyhedral sphere. An empty 2-dimensional face is called an ***empty polygon***. An ***empty pyramid*** of K is an induced subcomplex of K that consists of all the proper faces of a pyramid over a face of K.

CONNECTIVITY AND SUBCOMPLEXES

THEOREM 19.5.1 *Grünbaum* [Grü65]

The i-skeleton of every d-polytope contains a subdivision of $\mathrm{skel}_i(\Delta^d)$, *the i-skeleton of a d-simplex.*

THEOREM 19.5.2 *Folklore*

(i) *For $i > 0$,* $\mathrm{skel}_i(P)$ *is strongly connected.*

(ii) *For every face F, let $U_i(F)$ be the set of all i-faces of P containing F. Then if $i > \dim F$, $U_i(F)$ is strongly connected.*

Part (ii) follows at once from the fact that the faces of P containing F correspond to faces of the quotient polytope P/F. However, properties (i) and (ii) together are surprisingly strong, and all the known upper bounds for diameters of graphs of polytopes rely only on properties (i) and (ii) for the dual polytope.

THEOREM 19.5.3 *van Kampen and Flores* [Kam32, Flo32, Wu65]

For $i \geq \lfloor d/2 \rfloor$, $\mathrm{skel}_i(\Delta^{d+1})$ *is not embeddable in S^{d-1} (and hence not in the boundary complex of any d-polytope).*

(This extends the fact that K_5 is not planar.)

A beautiful extension of Balinski's theorem was offered by Lockenberg:

CONJECTURE 19.5.4 *Lockeberg*

For every partition of $d = d_1 + d_2 + \cdots + d_k$ and two vertices v and w of P, there are k disjoint paths between v and w such that the ith path is a path of d_i-faces in which any two consecutive faces have $(d_i - 1)$-dimensional intersection.

(Here by "disjoint" we do not refer to the common first vertex v and last vertex w.)

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SHELLABILITY AND COLLAPSIBILITY

THEOREM 19.5.5 *Bruggesser and Mani* [BM71]

Boundary complexes of polytopes are shellable.

The proof of Bruggesser and Mani is based on starting with a point near the center of a facet and moving from this point to infinity, and back from the other direction, keeping track of the order in which facets are seen. This proves a stronger form of shellability, in which each K_i is the complex spanned by all the facets that can be seen from a particular point in $\mathbb{R}^d$. It follows from shellability that

THEOREM 19.5.6

Polytopes are collapsible.

On the other hand,

THEOREM 19.5.7 *Ziegler* [Zie98]

There are d-polytopes, $d \geq 4$, whose boundary complexes are not extendably shellable.

THEOREM 19.5.8

There are triangulations of the $(d-1)$-sphere that are not shellable.

Lickorish [Lic91] produced explicit examples of nonshellable triangulations of S^3. His result was that a triangulation containing a sufficiently complicated knotted triangle was not shellable. Hachimori and Ziegler [HZ00] produced simple examples and showed that a triangulation containing any knotted triangle is not "constructible," constructibility being a strictly weaker notion than shellability. For more on shellability, see [DK78, Bjö92].

FACET-FORMING POLYTOPES AND "SMALL" LOW-DIMENSIONAL FACES

THEOREM 19.5.9 *Perles and Shephard* [PS67]

Let P be a d-polytope such that the maximum number of k-faces of P on any $(d-2)$-sphere in the skeleton of P is at most $(d-1-k)/(d+1-k)f_k(P)$. Then P is a nonfacet.

An example of a nonfacet that is simple was found by Barnette [Bar69]. Some of the proofs of Perles and Shephard use metric properties of polytopes, and for a few of the results alternative proofs using shellability were found by Barnette [Bar80].

THEOREM 19.5.10 *Schulte* [Sch85]

The cuboctahedron and the icosidodecahedron are nonfacets.

PROBLEM 19.5.11

Is the icosahedron facet-forming?

For all other regular polytopes the situation is known. The simplices and cubes in any dimension and the 3-dimensional octahedron are facet-forming. All other regular polytopes with the exception of the icosahedron are known to be nonfacets.

It is very interesting to see what can be said about metric properties of facets (or of low-dimensional faces) of a convex polytope.

THEOREM 19.5.12 *Bárány* (unpublished)

There is an $\epsilon > 0$ such that every d-polytope, $d > 2$, has a facet F for which no balls B_1 of radius R and B_2 of radius $(1+\epsilon)R$ satisfy $B_1 \subset F \subset B_2$.

The stronger statement where balls are replaced by ellipses is open.

Next, we try to understand if it is possible for all the k-faces of a d-polytope to be isomorphic to a given polytope P. The following conjecture asserts that if d is large with respect to k, this can happen only if P is either a simplex or a cube.

CONJECTURE 19.5.13 *Kalai* [Kal90]

For every k there is a $d(k)$ such that every d-polytope with $d > d(k)$ has a k-face that is either a simplex or combinatorially isomorphic to a k-dimensional cube.

Julian Pfeifle showed on the basis of the Wythoff construction (see Chapter 18), that $d(k) > (2k-1)(k-1)$, for $k \geq 3$.

For simple polytopes, it follows from the next theorem that if $d > ck^2$ then every d-polytope has a k-face F such that $f_r(F) \leq f_r(C_k)$. (Here, C_k denotes the k-dimensional cube.)

THEOREM 19.5.14 *Nikulin* [Nik86]

The average number of r-dimensional faces of a k-dimensional face of a simple d-dimensional polytope is at most

$$\binom{d-r}{d-k} \cdot \left(\left(\binom{\lfloor d/2 \rfloor}{r} + \binom{\lfloor (d+1)/2 \rfloor}{r} \right) \Big/ \left(\binom{\lfloor d/2 \rfloor}{k} + \binom{\lfloor (d+1)/2 \rfloor}{k} \right) \right).$$

Nikulin's theorem appeared in his study of reflection groups in hyperbolic spaces. The existence of reflection groups of certain types implies some combinatorial conditions on their fundamental regions (which are polytopes), and Vinberg [Vin85], Nikulin [Nik86], Khovanski [Kho86], and others showed that in high dimensions these combinatorial conditions lead to a contradiction. There are still many open problems in this direction: in particular, to narrow the gap between the dimensions above for which those reflection groups cannot exist and the dimensions for which such groups can be constructed.

THEOREM 19.5.15 *Kalai* [Kal90]

Every d-polytope for $d \geq 5$ has a 2-face with at most 4 vertices.

THEOREM 19.5.16 *Meisinger, Kleinschmidt, and Kalai* [MKK00]

Every (rational) d-polytope for $d \geq 9$ has a 3-face with at most 77 2-faces.

The last two theorems and the next one are proved using the linear inequalities for flag numbers that are known via intersection homology of toric varieties (toric h-vectors); see Chapter 17. Those inequalities were known to hold only for rational polytopes but the work of Karu [Kar04] extended them for non-rational polytopes as well.

The 120-cell shows that there are 4-polytopes all whose 2-faces are pentagons.

CONJECTURE 19.5.17 *Pack*

Every simple 4-polytope without a 2-face with at most 4 vertices has at least 600 vertices.

This conjecture may apply also to general polytopes. It is related in spirit to Theorem 17.5.10 by Blind and Blind identifying the cube as the "smallest" polytope with no triangular 2-faces. The conjecture may also apply to duals of arbitrary triangulations of S^3. However, it does not apply to (duals of) triangulated homology 3-sphere! Lutz, Sulanke, and Sullivan constructed a triangulation of Poincarés dodecahedral sphere that has only 18 vertices and 100 facets and has the property that every edge belongs to at least five simplices.

Here are two closely related questions about 4-polytopes:

PROBLEM 19.5.18 *Ziegler*

What is the maximal number of 2-faces which are not 4-gons for a 4-polytope with n facets?

PROBLEM 19.5.19 *Nevo, Santos, and Wilson [NSW16]*

What is the maximal number of facets which are not simplices for a 4-polytope with n vertices?

Nevo, Santos and Wilson [NSW16] gave lower bounds of $\Omega(n^{3/2})$ for both these questions.

Another possible extension of Theorem 19.5.15 is

CONJECTURE 19.5.20 *For every $\epsilon > 0$, every d-polytope for $d \geq d(\epsilon)$ has an edge such that more than a fraction $(1-\epsilon)$ of 2-faces containing it has at most 4 vertices.*

This conjecture is motivated by recent results (related to the Gallai-Sylvester Theorem) [DSW14] showing it to hold for zonotopes with $f(\epsilon) = 12/\epsilon$.

QUOTIENTS AND DUALITY

We talked about k-faces of d-polytopes, and one can also study, in a similar fashion, quotients of polytopes.

CONJECTURE 19.5.21 *Perles*

For every k there is a $d'(k)$ such that every d-polytope with $d > d'(k)$ has a k-dimensional quotient that is a simplex.

As was mentioned in the first section, $d'(2) = 3$. The 24-cell, which is a regular 4-polytope all of whose faces are octahedra, shows that $d'(3) > 4$.

THEOREM 19.5.22 *Meisinger, Kleinschmidt, and Kalai* [MKK00]

Every d-polytope with $d \geq 9$ has a 3-dimensional quotient that is a simplex.

Of course, Conjecture 19.5.13 implies Conjecture 19.5.21. Another stronger form of Conjecture 19.5.21, raised in [MKK00], is whether, for every k, a high-enough dimensional polytope must contain a face that is a k-simplex, or its polar dual must contain such a face. Adiprasito [Adi11] showed that for the analogous question for polytopal spheres, the answer is negative.

PROBLEM 19.5.23

For which values of k and r are there d-polytopes other than the d-simplex that are both k-simplicial and r-simple?

It is known that this can happen only when $k+r \leq d$. There are infinite families of $(d-2)$-simplicial and 2-simple polytopes, and some examples of $(d-3)$-simplicial and 3-simple d-polytopes.

Concerning this problem Peter McMullen recently noted that the polytopes r_{st}, discussed in Coxeter's classic book on regular polytopes [Cox63] in Sections 11.8 and 11.x, are $(r+2)$-simplicial and $(d-r-2)$-simple, where $d = r+s+t+1$. These so-called ***Gosset-Elte polytopes*** arise by the Wythoff construction from the finite reflection groups (see Chapter 18 of this Handbook); we obtain a finite polytope whenever the reflection group generated by the Coxeter diagram with r, s, t nodes on the three arms is finite, that is, when

$$1/(r+1) + 1/(s+1) + 1/(t+1) > 1.$$

The largest exceptional example, 2_{41}, is related to the Weyl group E_8. The Gosset-Elte polytope 2_{41} is a 4-simple 4-simplicial 8-polytope with 2160 vertices. Are there 5-simplicial 5-simple 10-polytopes?

PROBLEM 19.5.24

For which values of k and d are there self-dual k-simplicial d-polytopes other than the d-simplex?

THEOREM 19.5.25

For $d > 2$, there is no cubical d-polytope P whose dual is also cubical.

I am not aware of a reference for this result but it can easily be proved by exhibiting a covering map from the standard cubical complex realizing $\mathbb{R}^{d-1}$ into the boundary complex of P.

We have considered the problem of finding very special polytopes as "subobjects" (faces, quotients) of arbitrary polytopes. What about realizing arbitrary polytopes as "subobjects" of very special polytopes? There was an old conjecture that every polytope can be realized as a subpolytope (namely the convex hull of a subset of the vertices) of a stacked polytope. However, this conjecture was refuted by Adiprasito and Padrol [AP14]. Perles and Sturmfels asked whether every simplicial d-polytope can be realized as the quotient of some neighborly even-dimensional polytope. (Recall that a $2m$-polytope is ***neighborly*** if every m vertices are the vertices of an $(m-1)$-dimensional face.) Kortenkamp [Kor97] proved that this is the case for d-polytopes with at most $d+4$ vertices. For general polytopes, "neighborly polytopes" should be replaced here by "weakly neighborly" polytopes, introduced by Bayer [Bay93], which are defined by the property that every set of k vertices is contained in a face of dimension at most $2k-1$. The only theorem of this flavor I am aware of is by Billera and Sarangarajan [BS96], who proved that every $(0,1)$-polytope is a face of a traveling salesman polytope.

RECONSTRUCTION

THEOREM 19.5.26 *An extension of Whitney's theorem* [Grü67]

d-polytopes are determined by their $(d-2)$-skeletons.

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THEOREM 19.5.27 *Perles* (unpublished, 1973)

Simplicial d-polytopes are determined by their $\lfloor d/2 \rfloor$-skeletons.

This follows from the following theorem (here, $\mathrm{ast}(F,P)$ is the complex formed by the faces of P that are disjoint to all vertices in F).

THEOREM 19.5.28 *Perles* (1973)

Let P be a simplicial d-polytope.

(i) *If F is a k-face of P, then* $\mathrm{skel}_{d-k-2}(\mathrm{ast}(F,P))$ *is contractible in* $\mathrm{skel}_{d-k-1}(\mathrm{ast}(F,P))$.

(ii) *If F is an empty k-simplex, then* $\mathrm{ast}(F,P)$ *is homotopically equivalent to* S^{d-k}*; hence,* $\mathrm{skel}_{d-k-2}(\mathrm{ast}(F,P))$ *is not contractible in* $\mathrm{skel}_{d-k-1}(\mathrm{ast}(F,P))$.

An extension of Perles's theorem for manifolds with vanishing middle homology was proved by Dancis [Dan84].

THEOREM 19.5.29 *Blind and Mani-Levitska* [BM87]

Simple polytopes are determined by their graphs.

Blind and Mani-Levitska described their theorem in a dual form and considered $(d-1)$-dimensional "puzzles" whose pieces are simplices and we wish to reconstruct the puzzle based on the "local" information of which two simplices share a facet. Joswig extended their result to more general puzzles where the pieces are general $(d-1)$-dimensional polytopes, and the way in which every two pieces sharing a facet are connected is also prescribed. A simple proof is given in [Kal88a]. This proof also shows that k-dimensional skeletons of simplicial polytopes are also determined by their "puzzle." When this is combined with Perles's theorem it follows that:

THEOREM 19.5.30 *Kalai and Perles*

Simplicial d-polytopes are determined by the incidence relations between i- and $(i+1)$-faces for every $i > \lfloor d/2 \rfloor$.

CONJECTURE 19.5.31 *Haase and Ziegler*

Let G be the graph of a simple 4-polytope. Let H be an induced, nonseparating, 3-regular, 3-connected planar subgraph of G. Then H is the graph of a facet of P.

Haase and Ziegler [HZ02] showed that this is not the case if H is not planar. Their proof touches on the issue of embedding knots in the skeletons of 4-polytopes.

PROBLEM 19.5.32

Are simplicial spheres determined by the incidence relations between their facets and subfacets?

THEOREM 19.5.33 *Björner, Edelman, and Ziegler* [BEZ90]

Zonotopes are determined by their graphs.

THEOREM 19.5.34 *Babson, Finschi, and Fukuda* [BFF01]

Duals of cubical zonotopes are determined by their graphs.

In all instances of the above theorems except the single case of the theorem of Blind and Mani-Levitska, the proofs give reconstruction algorithms that are polynomial in the data. It was an open question if a polynomial algorithm exists to determine a simple polytope from its graph. A polynomial "certificate" for reconstruction was recently found by Joswig, Kaibel, and Körner [JKK02]. Finally, Friedman [Fri09] proved

THEOREM 19.5.35 *Friedman* [Fri09]

There is a polynomial type algorithm to determine a simple d-polytope (up to combinatorial isomorphism) from its graph.

An interesting problem was whether there is an e-dimensional polytope other than the d-cube with the same graph as the d-cube.

THEOREM 19.5.36 *Joswig and Ziegler* [JZ00]

For every $d \geq e \geq 4$ there is an e-dimensional cubical polytope with 2^d vertices whose $(\lfloor e/2 \rfloor - 1)$-skeleton is combinatorially isomorphic to the $(\lfloor e/2 \rfloor - 1)$-skeleton of a d-dimensional cube.

Earlier, Babson, Billera, and Chan [BBC97] found such a construction for cubical spheres.

Another issue of reconstruction for polytopes that was studied extensively is the following: In which cases does the combinatorial structure of a polytope determine its geometric structure (up to projective transformations)? Such polytopes are called ***projectively unique***. McMullen [McM76] constructed projectively unique d-polytopes with $3^{d/3}$ vertices. The major unsolved problem whether there are only finitely many projectively unique polytopes in each dimension was settled in

THEOREM 19.5.37 *Adiprasito and Ziegler* [AZ15]

There is an infinite family of projectively unique 69-dimensional polytopes.

The key for this result was the construction of an infinite family of 4-polytopes whose realization spaces have bounded (at most 96) dimension.

A related question is to understand to what extent we can reconstruct the internal structure of a polytope, namely the combinatorial types of subpolytopes (or, equivalently, the oriented matroid described by the vertices) from its combinatorial structure.

THEOREM 19.5.38 *Shemer [She82]*

Neighborly even-dimensional polytopes determine their internal structure.

EMPTY FACES AND POLYTOPES WITH FEW VERTICES

THEOREM 19.5.39 *Perles* (unpublished, 1970)

Let $f(d,k,b)$ be the number of combinatorial types of k-skeletons of d-polytopes with $d+b+1$ vertices. Then, for fixed b and k, $f(d,k,b)$ is bounded.

This follows from

THEOREM 19.5.40 *Perles* (unpublished, 1970)

The number of empty i-pyramids for d-polytopes with $d+b$ vertices is bounded by a function of i and b.

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For another proof of this theorem see [Kal94].

Here is a beautiful recent subsequent result:

THEOREM 19.5.41 *Padrol* [Pad16]

The number of d-polytopes with $d+b$ vertices and $d+c$ facets is bounded by a function of b and c.

EMPTY FACE NUMBERS AND RELATED BETTI NUMBERS

For a d-polytope P, let $e_i(P)$ denote the number of empty i-simplices of P.

PROBLEM 19.5.42

Characterize the sequence of numbers $(e_1(P), e_2(P), \ldots, e_d(P))$ arising from simplicial d-polytopes and from general d-polytopes.

The following theorem, which was motivated by commutative-algebraic concerns, confirmed a conjecture by Kleinschmidt, Kalai, and Lee [Kal94].

THEOREM 19.5.43 *Migliore and Nagel* [MN03, Nag08]

For all simplicial d-polytopes with prescribed h-vector $h = (h_0, h_1, \ldots, h_d)$, the number of i-dimensional empty simplices is maximized by the Billera-Lee polytopes $P_{BL}(h)$.

$P_{BL}(h)$ is the polytope constructed by Billera and Lee [BL81] (see Chapter 17) in their proof of the sufficiency part of the g-theorem. Migliore and Nagel proved that for a prescribed f-vector, the Billera-Lee polytopes maximize even more general parameters that arise in commutative algebra: the sum of the ith Betti numbers of induced subcomplexes on j vertices for every i and j. (These sums correspond to Betti numbers of the resolution of the Stanley-Reisner ring associated with the polytope.) The case $j = i + 2$ reduces to counting empty faces. It is quite possible that the theorem of Migliore and Nagel extends to general simplicial spheres with prescribed h-vector and to general polytopes with prescribed (toric) h-vector. (However, it is not yet known in these cases that the h-vectors are always those of Billera-Lee polytopes; see Chapter 17.)

Recently, valuable connections between these Betti numbers, discrete Morse theory, face numbers, high-notions of chordality, and metrical properties of polytopes are emerging [ANS16, Bag16].

STANLEY-REISNER RINGS

An algebraic object associated to simplicial polytopes, triangulated spheres, and general simplicial complexes is the Stanley-Reisner ring which gives crucial information on face numbers (discussed in Chapter 17). The Stanley-Reisner ring also has applications to the study of skeletons of polytopes discussed in this chapter.

For every simplicial d-polytope P with n vertices one can associate an ideal $I(P)$ of monomials in $n-d$ variables with $h_i(P)$ degree-i monomials for $i = 1, 2, \ldots, d$. (Moreover, $I(P)$ is *shifted* [Kal02, Kal94]). This construction is based on deep properties of the Stanley-Rieser ring associated to simplicial d polytopes which conjecturally extends to arbitrary triangulations of S^{d-1}. The shifted ideal $I(P)$

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carries important structural properties of P and its skeletons. For example, there are connections with rigidity [Lee94, Kal94]. Moving from an arbitrary shifted order ideal of monomials to a triangulation is described in [Kal88b] and is the basis for Theorem 19.6.5. There are extensions and analogues of the Stanley-Reisner rings for nonsimplicial polytopes, for restricted classes of polytopes and for other cellular objects.

19.6 COUNTING POLYTOPES, SPHERES AND THEIR SKELETA

THEOREM 19.6.1 *There are only a "few" simplicial d-polytopes with n facets Durhuus and Jonsson: Benedetti and Ziegler* [DJ95, BZ11]

The number of distinct (isomorphism types) simplicial d-polytopes with n facets is at most C_d^n, where C_d is a constant depending on d.

CONJECTURE 19.6.2 *There are only a "few" triangulations of S^d with n facets*

The number of distinct (isomorphism types) triangulations of S^{d-1} with n facets is at most C_d^n, where C_d is a constant depending on d.

See Gromov's paper [Gro00] for some discussion.

CONJECTURE 19.6.3 *There are only a "few" graphs of polytopes*

The number of distinct (isomorphism types) of graphs of d-polytopes with n vertices is at most C_d^n, where C_d is a constant depending on d.

Both conjectures are open already for $d = 4$. As far as we know, it is even possible that the same constant applies for all dimensions.

We can also ask for the number of simplicial d-polytopes and triangulation of spheres with n vertices. Let $s(d, n)$ denotes the number of triangulations of S^{d-1} with n vertices and let $p_s(d, n)$ be the number of combinatorial types of simplicial d-polytopes with n vertices, and let $p(d, n)$ be the number of combinatorial types of (general) d-polytopes with n vertices.

THEOREM 19.6.4 *Goodman and Pollack; [GP86] Alon [Alo86]*

For some absolute constant C, $\log p_s(d, n) \leq \log p(d, n) \leq Cd^2 n \log n$.

THEOREM 19.6.5 *Kalai [Kal88b]*

For a fixed d, $\log s(d, n) = \Omega(n^{[d/2]})$

THEOREM 19.6.6 *Nevo, Santos, and Wilson [NSW16]*

Let k be fixed. Then $\log s(2k + 1, n) = \Omega(n^{k+1})$.

The case of triangulations of 3-spheres is of particular interest. The quadratic lower bound for $\log s(3, n)$ improves an earlier construction from [PZ04]. Stanley's upper bound theorem for triangulated spheres (see Chapter 17) implies that $\log s(d, n) = \Omega(n^{[(k+1)/2]\log n})$.

We already mentioned that there is a rich enumerative theory for 3-polytope. Perles found formulas for the number of d-polytopes, and simplicial d-polytopes with $d+3$ vertices. His work used Gale's diagram and is presented in Grünbaum's classical book on polytopes [Grü67]. There is also a formula for the number of stacked d-polytopes with n labelled vertices [BP71].

19.7 CONCLUDING REMARKS AND EXTENSIONS TO MORE GENERAL OBJECTS

The reader who compares this chapter with other chapters on convex polytopes may notice the sporadic nature of the results and problems described here. Indeed, it seems that our main limits in understanding the combinatorial structure of polytopes still lie in our ability to raise the right questions. Another feature that comes to mind (and is not unique to this area) is the lack of examples, methods of constructing them, and means of classifying them.

We have considered mainly properties of general polytopes and of simple or simplicial polytopes. There are many classes of polytopes that are either of intrinsic interest from the combinatorial theory of polytopes, or that arise in various other fields, for which the problems described in this chapter are interesting.

Most of the results of this chapter extend to much more general objects than convex polytopes. Finding combinatorial settings for which these results hold is an interesting and fruitful area. On the other hand, the results described here are not sufficient to distinguish polytopes from larger classes of polyhedral spheres, and finding delicate combinatorial properties that distinguish polytopes is an important area of research. Few of the results on skeletons of polytopes extend to skeletons of other convex bodies [LR70, LR71, GL81], and relating the combinatorial theory of polytopes with other aspects of convexity is a great challenge.

19.8 SOURCES AND RELATED MATERIAL

FURTHER READING

Grünbaum [Grü75] is a survey on polytopal graphs and many results and further references can be found there. More material on the topic of this chapter and further relevant references can also be found in [Grü67, Zie95, BMSW94, KK95, BL93, DRS10]. Several chapters of [BMSW94] are relevant to the topic of this chapter: The authors chapter [Kal94] expands on various topics discussed here and in Chapter 17. Martini's chapter on the regularity properties of polytopes contains further references on facet-forming polytopes and nonfacets. The original papers on facet-forming polytopes and nonfacets contain many more results, and describe relations to questions on tiling spaces with polyhedra.

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RELATED CHAPTERS

Chapter 15: Basic properties of convex polytopes
Chapter 17: Face numbers of polytopes and complexes

Chapter 17 discusses f-vectors of polytopes (and more general cellular structures), and related parameters such as h-vectors and g-vectors for simplicial polytopes, and flag-f-vectors, toric h-vectors, toric g-vectors and the cd-index of general polytopes. There are many relations between these parameters and the combinatorial study of graphs and skeleta of polytopes.

Chapter 49: Linear programming
Chapters 7, 16, 18, 20, and 61 are also related to some parts of this chapter.

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