LETTER

Causality Violating Geodesics in Bonnor's Rotating Dust Metric

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We exhibit timelike geodesic paths for a metric, introduced by Bonnor [11] and considered also by Steadman [13], and show that coordinate time runs backward along a portion of these geodesics.

1. INTRODUCTION

Causality violating paths are known to exist in many spacetimes satisfying the Einstein field equations, especially those solutions associated with rotating matter or rotating singularities [1–3] (see also Bonnor [4], and references in [5]). However, examples of causality violating geodesics are sparse in the scientific literature. Chandrasekhar and Wright [6] showed that the Gödel spacetime [1] admits no closed timelike geodesics, even though the spacetime includes closed timelike curves. In the case of the Kerr metric, Carter [3] proved that closed timelike paths exist for some parameter values of the metric. Calvani *et al*. [7] then showed that a class of timelike geodesics in the Kerr spacetime do not violate causality, but de Felice and Calvani [8] later found numerical solutions for null geodesics that

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violate causality under "very particular conditions." More recently, Steadman [9] found closed, circular timelike geodesics in the van Stockum spacetime [10] for a rotating infinite dust cylinder, along which coordinate time runs backward.

In this paper, we consider Bonnor's axially symmetric rotating dust cloud metric [11], on which a closed (but nongeodesic) null path has already been observed to exist [12]. We analyze the causality violating region and give numerical plots of timelike geodesics along which coordinate time runs backward on portions of the paths. We show that some of the geodesics connect the causality violating region to asymptotically flat spatial infinity.

In Section 2, we introduce notation and describe the Bonnor dust cloud metric. In Section 3, we prove that the causality violating region for this spacetime includes the interior of a torus centered at the origin of coordinates, and we use that information to produce numerical plots of causality violating geodesics. Concluding remarks are given Section 4. Finally in the Appendix, we prove that no timelike or null circular closed geodesics, centered on the axis of symmetry of the spacetime (analogous to geodesics considered by Steadman [9]), violate causality.

2. BONNOR'S DUST METRIC

A solution to the field equations given by Bonnor [11] is an axially symmetric metric which describes a cloud of rigidly rotating dust particles moving along circular geodesics about the z-axis in hypersurfaces of $z =$ constant. The line element is

$$ds^2 = -dt^2 + (r^2 - n^2) \, d\phi^2 + 2n \, dt \, d\phi + e^\mu (dr^2 + dz^2), \qquad (1)$$

where, in Bonnor's comoving (i.e., corotating) coordinates,

$$n = \frac{2hr^2}{R^3}, \quad \mu = \frac{h^2 r^2 (r^2 - 8z^2)}{2R^8}, \quad R^2 = r^2 + z^2, \qquad (2)$$

h is a rotation parameter, and we have the gauge condition

$$(g_{t\phi})^2 - g_{tt} g_{\phi\phi} = r^2. \qquad (3)$$

Bonnor's metric has an isolated singularity at $r = z = 0$. The energy density ρ is given by

$$8\pi\rho = \frac{4e^{-\mu} h^2 (r^2 + 4z^2)}{R^8}. \qquad (4)$$

As $R \to \infty$, ρ approaches zero rapidly and the metric coefficients tend to Minkowski values. Moreover, all the Riemann curvature tensor elements vanish at spatial infinity.

3. CAUSALITY VIOLATING TIMELIKE GEODESICS

From Eqs. (1) and (2) we obtain the Hamiltonian

$$H = \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta,$$

$$= \frac{(n^2 - r^2)}{2r^2} p_t^2 + \frac{1}{2r^2} p_\phi^2 + \frac{n}{r^2} p_t p_\phi + \frac{1}{2} e^{-\mu} \left(p_r^2 + p_z^2 \right), \tag{5}$$

where the angular momentum p_ϕ and the energy $E = -p_t$ are conserved quantities. Since for timelike geodesics $H = -1/2$, it follows from Eq. (5) that timelike geodesics with angular momentum p_ϕ and energy E can exist only in the region S_B of spacetime given by

$$S_B = \{(t, \phi, r, z)| - p_\phi^2 + 2nEp_\phi + (r^2 - n^2)E^2 - r^2 \geq 0\}. \tag{6}$$

At any point in S_B, a geodesic with initial timelike momentum, (E, p_ϕ, p_r, p_z) exists with p_r and p_z chosen so that $r^2 e^{-\mu}(p_r^2 + p_z^2)$ is equal to the left side of the inequality in (6).

Remark: Steadman [13] investigated the "allowed region" for null geodesics in Bonnor's metric. He showed that the region corresponding to S_B, for null geodesics, is topologically connected for $p_\phi/E \leq 2\sqrt{2h}$ and has two components otherwise. So for $p_\phi/E > 2\sqrt{2h}$, null geodesics may be trapped in Steadman's "central region of confinement," i.e., in the component with smaller r values. We find that the situation is qualitatively the same for timelike geodesics, except that in this case the topology of the physically allowed region depends on both p_ϕ and E and not just their ratio.

To find causality violating timelike geodesics, we consider Hamilton's equation for $\dot{t}$ for an arbitrary geodesic, where the overdot represents differentiation with respect to proper time, and for convenience, we set $h = 1$.

$$\dot{t} = E + \frac{2p_\phi}{(r^2 + z^2)^{3/2}} - \frac{4Er^2}{(r^2 + z^2)^3} \tag{7}$$

The following proposition shows that a causality violating region exists for this spacetime.

Proposition 1: *There exists a spacelike torus T centered at $r = z = 0$ in the Bonnor spacetime such that $\dot{t} < 0$ along any timelike geodesic in the interior of T.*

Proof. In the plane $z = 0$, Eq. (7) becomes,

$$\dot{t} = \frac{2p_\phi}{r^3} - \frac{E(4 - r^4)}{r^4}.$$

For $z = 0$ and $\dot{t} < 0$, it follows that we must have

$$-[(r^4 - 4)E + 2r p_\phi] > 0. \tag{8}$$

For given values of E and p_ϕ, it follows from Eq (5) that $H = -1/2$ if and only if

$$p_r^2 + p_z^2 = \frac{e^\mu}{r^2}[(E^2 - 1)r^2 - (nE - p_\phi)^2]. \tag{9}$$

Solutions to (9) exist provided that,

$$(E^2 - 1)r^2 - (nE - p_\phi)^2 \geq 0. \tag{10}$$

Now substituting $z = 0$, $h = 1$ and Eqs. (2) into (10), and multiplying by r^2 yields,

$$(E^2 - 1)r^4 - (2E - r p_\phi)^2 \geq 0. \tag{11}$$

It is easy to check that the solution set to inequalities (8) and (11) is nonempty with strict inequality in (11) (see the numerical examples below). It follows from standard continuity arguments that the simultaneous solution set to inequalities $\dot{t} < 0$ and (10) contains a nonempty open set in (r, ϕ, z, E, p_ϕ) space, which is independent of ϕ. Thus, $\dot{t} < 0$ in a nonempty open subset of S_B which is independent of ϕ and t (c.f. Eq. (5)), which therefore contains a torus T as described above. This completes the proof.

Using the Proposition and the Remark immediately above, we now give two numerical examples of timelike causality violating geodesics (CVG), both lying in the $z = 0$ plane with $p_z = 0$. One is a spatially bound, evidently quasiperiodic geodesic, and the other is an unbound geodesic. In the quasiperiodic example (Figures 1, 2, and 3) we let $E = 1.3$, $p_\phi = 3$, then inequalities (9) and (11) are both
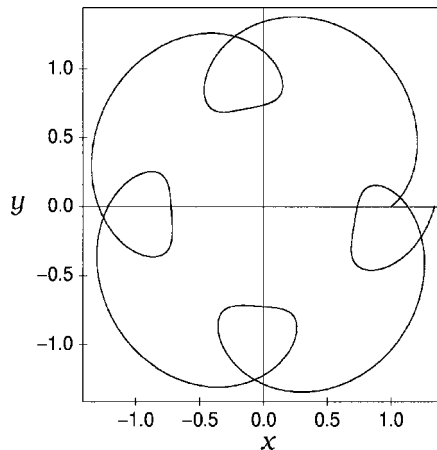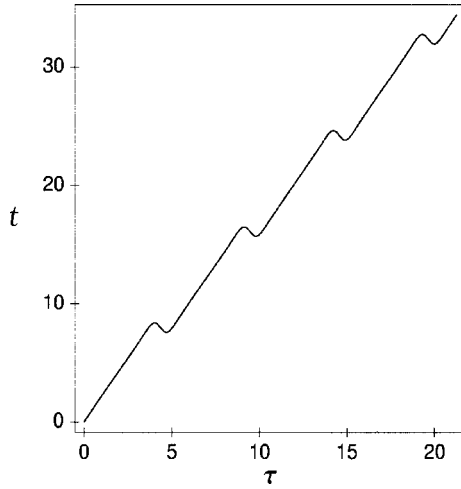

Figure 1. Quasiperiodic CVG.

Figure 2. t vs. τ.

satisfied for $r \in [0.722235\ldots, 0.784571\ldots)$. The initial values for the timelike CVG of Figure 1 are $r = 1$, $p_r = 0.934785\ldots$. For the unbound CVG example (Figures 4, 5, and 6) we have $E = 4.12078\ldots$, $p_\phi = 2.5$, then inequalities (9) and (11) are both satisfied for $r \in [1.15680\ldots, 1.25462\ldots)$. The initial values for the timelike CVG of Figure 4 are $r = 4$, $p_r = -4$.
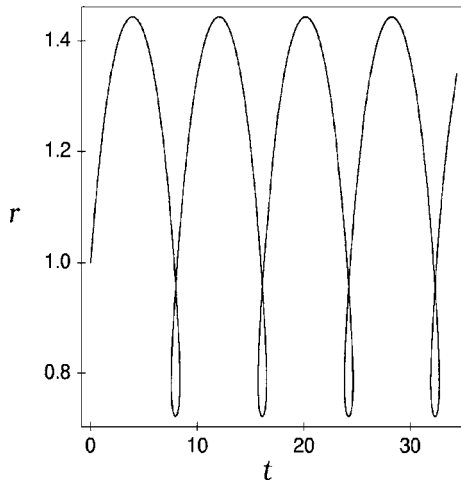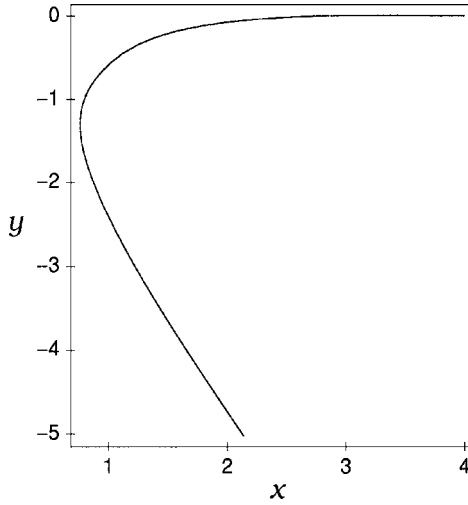

Figure 3. r vs. t.

Figure 4. Unbound CVG.

4. CONCLUDING REMARKS

Bonnor's dust metric includes a causality violating region with nonempty interior. Coordinate time runs backward along the portions of all timelike geodesics
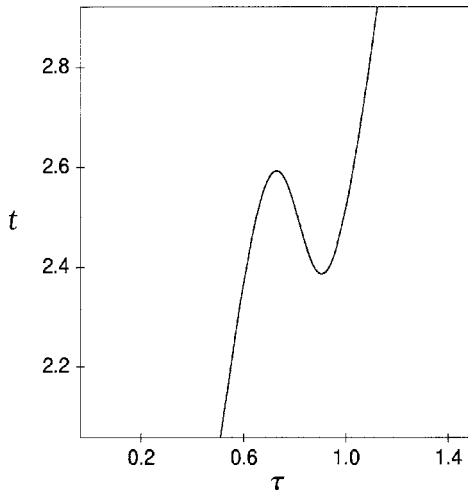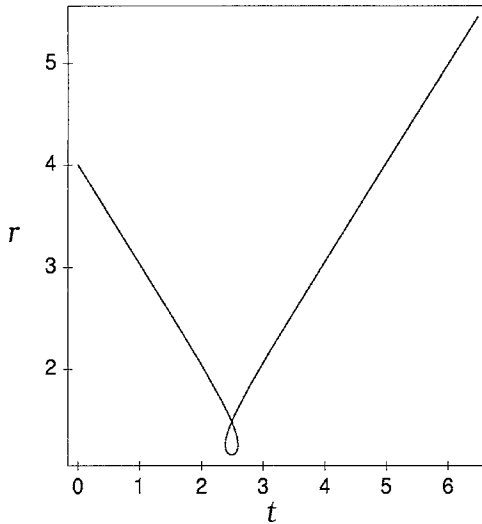

Figure 5. t vs. τ.

Figure 6. r vs. t.

within this region. Bonnor [14] has argued that closed timelike paths are possible, not only in physically unrealistic solutions to the Einstein field equations, but also for solutions that "refer to ordinary materials in situations which might occur in the laboratory, or in astrophysics." He has called for new interpretations of this phenomenon. The spacetime considered here has some unrealistic features. It has an isolated singularity with no event horizon. However, the singularity is inaccessible to all null geodesics except possibly along the axis of symmetry of the spacetime, a set of zero measure. In addition, the metric coefficients for this spacetime tend to Minkowski values at spatial infinity and all the Riemann curvature tensor elements vanish there. Numerical calculations show that the causality violating region of this spacetime is accessible to timelike geodesic paths from distant, asymptotically flat regions of the spacetime. Thus the causality violating region may be reached by an observer with no expenditure of energy. A new physical interpretation of closed timelike curves may have interesting implications for this spacetime.

APPENDIX

In this Appendix we show that Bonnor's dust metric does not have timelike or null circular geodesics about the z-axis in hypersurfaces of z = constant with $\dot{t} \leq 0$. We set the parameter $h = 1$ and shall assume that $E > 0$. Hamilton's

equation for $\dot{t}$ is Eq. (7). Hamilton's equation for $\dot{\phi}$ is:

$$\dot{\phi} = \frac{p_\phi}{r^2} - \frac{2E}{(r^2 + z^2)^{3/2}} \tag{12}$$

If we let $p_r = p_z = 0$ in Hamilton's equations for $\dot{p}_r$ and $\dot{p}_z$, we have

$$\dot{p}_r = \frac{p_\phi^2}{r^3} - \frac{6Ep_\phi r}{(r^2 + z^2)^{5/2}} - \frac{E^2(4 - 3(r^2 + z^2)^2)r}{(r^2 + z^2)^3}$$

$$+ \frac{3E^2(4r^2 - (r^2 + z^2)^3)r}{(r^2 + z^2)^4}, \tag{13}$$

$$\dot{p}_z = -\frac{6Ep_\phi z}{(r^2 + z^2)^{5/2}} + \frac{3E^2 z}{r^2 + z^2} + \frac{3E^2(4r^2 - (r^2 + z^2)^3)z}{(r^2 + z^2)^4}. \tag{14}$$

Equating the above equations to zero and solving for r and z we obtain the following solutions for z real and $r > 0$:

$$z = 0, \quad r = \frac{2E}{p_\phi}, \tag{15}$$

$$z = 0, \quad r = \frac{4E}{p_\phi}, \tag{16}$$

$$z = \pm \left(\left(\frac{2Er^2}{p_\phi} \right)^{\frac{2}{3}} - r^2 \right)^{\frac{1}{2}}. \tag{17}$$

Solution (17) reduces to solution (15) if we require that $z = 0$. The above solutions will be timelike circular geodesics provided $p_r = p_z = 0$. To check this we introduce the function K below,

$$e^\mu \left(H + \frac{1}{2} \right) = \frac{1}{2}(p_r^2 + p_z^2) + \frac{e^\mu}{2} K = 0, \tag{18}$$

where H is the Hamiltonian of Eq. (5), and using Eqs. (2), we have

$$K(r, z, E, p_\phi) = 1 + \frac{p_\phi^2}{r^2} - \frac{4Ep_\phi}{(r^2 + z^2)^{3/2}} + \left(\frac{4r^2}{(r^2 + z^2)^3} - 1 \right) E^2. \tag{19}$$

From Eq. (18) we see that for circular timelike geodesics K must vanish so that $p_r = p_z = 0$. Substituting solution (15) ($z = 0$) or solution (17) ($z \neq 0$) in K we find that $K = 0$ forces $E = 1$. Furthermore if we substitute solution (15) or (17) into Eq. (12) for $\dot{\phi}$ and Eq. (7) for $\dot{t}$, we get $\dot{\phi} = 0$ and $\dot{t} = E$, therefore the circular geodesics of solutions (15) and (17) correspond to the paths of Bonnor's dust particles. If we substitute solution (16) in K we find that

$E = ((1 + \sqrt{1 + (p_\phi/2)^4})/2)^{1/2}$ and this time Eqs. (12) and (7) give

$$\dot{\phi} = \frac{p_\phi^3}{32E^2}, \tag{20}$$

$$\dot{t} = E + \frac{p_\phi^4}{64E^2}. \tag{21}$$

We see from Eq. (21) that $\dot{t} > 0$ for these geodesics also.

Finally, repeating the above calculations for null geodesics we find that only solution (16) can be a null circular geodesic provided that $E = p_\phi/(2\sqrt{2})$ and in this case also $\dot{t} > 0$, where the overdot now represents differentiation with respect to an affine parameter.

ACKNOWLEDGMENTS

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