Problem of the Week 7, Spring 2006

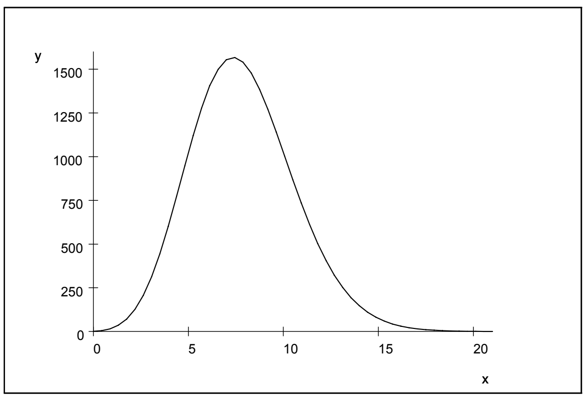
Solution by Andrew Jones. We want to maximize $a_1 \cdot a_2 \cdot ... \cdot a_n$, where the constraint is that $a_1 + a_2 + ... + a_n = 20$ and n is a positive integer.

1. First notice that all a_i are equal ($i \in (1, 2, ...n)$), since the greatest product of any values that add up to a certain number is when the values are equal.

2. Then notice that n must be between 1 and 20. If $n > 20$ then the as would be less than 1 which would lower the product (in fact it would be less than 1).

From (1) and (2) we have that the maximum looks like:

$$f(n) = \left(\frac{20}{n}\right)^n$$



Now we must find an n that maximizes f. Obviously not $n = 1$.
$\left(\frac{20}{2}\right)^2 = 100$
$\left(\frac{20}{3}\right)^3 = 296.296\,296\,3$
$\left(\frac{20}{4}\right)^4 = 625.0$
$\left(\frac{20}{5}\right)^5 = 1024.0$
$\left(\frac{20}{6}\right)^6 = 1371.742\,113$
$\left(\frac{20}{7}\right)^7 = 1554.260\,069 \quad * * * *$
$\left(\frac{20}{8}\right)^8 = 1525.878\,906$
$\left(\frac{20}{9}\right)^9 = 1321.561\,493$
as in the graph above the function is decreasing after $n = 9$. We notice that

$$\left(\frac{20}{7}\right)^7 = 1554.260\,069$$

is the maximum.