Problem of the Week 4, Spring 2006

Solution by Marc Missildine (edited). Let $p < q < r$ be prime numbers such that $pqr + 1$ is a perfect fourth power. We want to show that $(p+1)(q-1)(r-1)$ is also a fourth power. So assume that $pqr + 1 = a^4$ for some positive integer a. Thus

$$pqr = a^4 - 1 = (a^2 - 1)(a^2 + 1) = (a - 1)(a + 1)(a^2 + 1).$$

Since p, q, r are all prime and $p < q < r$ then $a^4 = pqr + 1 \geq 3 \cdot 5 \cdot 7 + 1 \geq 86$, that is $a > 3$ and so $a - 1 > 2$. Then we know that $a - 1, a + 1$, and $a^2 + 1$ are the only 3 prime factors of $a^4 - 1$. We can set $p = a - 1, q = a + 1$, and $r = a^2 + 1$, since $1 < a - 1 < a + 1 < a^2 + 1$ and $p < q < r$. Therefore $(p + 1)(q - 1)(r - 1) = (a - 1 + 1)(a + 1 - 1)(a^2 + 1 - 1) = a \cdot a \cdot a^2 = a^4$.