Problem of the Week 1, Spring 2006

Solution by Lucas Lembrick (edited). 945 is an odd abundant number. Note that $945 = 3^3 \cdot 5 \cdot 7$, so all divisors of 945 excluding 945 are: $1, 5, 7, 5\cdot 7, 3, 3\cdot 5, 3\cdot 7, 3\cdot 5\cdot 7, 3^2, 3^2\cdot 5, 3^2\cdot 7, 3^2\cdot 5\cdot 7, 3^3, 3^3\cdot 5, 3^3\cdot 7$. The sum of these numbers is 975.

The problem did not ask to prove that 945 is in fact the smallest odd abundant number. This is true and we (the organizers) are supplying a proof of this that does not require extensive calculations.

Proposition 1 *The number 945 is the smallest odd abundant number.*

Proof. One way to prove this is of course to calculate the sum of the divisors of all odd numbers less than 945 and verify that none of these sums exceeds their corresponding number. We take a better approach here. First note that if $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ is the factorization of n into different prime powers, then the sum of the divisors of n, denoted $\sigma(n)$ satisfies

$$\begin{aligned}\sigma(n) &= \left(1 + p_1 + p_1^2 + \cdots + p_1^{a_1}\right)\left(1 + p_2 + p_2^2 + \cdots + p_2^{a_2}\right)\cdots\left(1 + p_r + p_r^2 + \cdots + p_r^{a_r}\right) \\ &= \prod_{i=1}^{r}\left(1 + p_i + p_i^2 + \cdots + p_i^{a_i}\right) = \prod_{i=1}^{r}\frac{p_i^{a_i+1} - 1}{p_i - 1}.\end{aligned}$$

Now using this formula a number is abundant if and only if $\sigma(n) > 2n$. We will prove our proposition based on the following three lemmas:

Lemma 2 *If $n = p^a$ then n is not abundant.*

Lemma 3 *If $n = pqr$ (p, q, r distinct primes) and n is odd, then n is not abundant.*

Lemma 4 *If $n = 3^2 qr$ (q, r distinct primes at least 5), then n is not abundant.*

Lemma 5 *If $n = p^a q^b$ (p, q distinct primes) and n is odd, then n is not abundant.*

Assume the lemmas are true. Let n be an odd positive integer. If n has four or more different prime factors then $n > 3\cdot 5\cdot 7\cdot 11 = 1155$. If n has exactly one or two different prime factors then, by Lemmas 2 and 5, n is not abundant. So we may assume n has exactly three different prime factors and moreover, by Lemma 3, we may assume at least one of these factors has a power greater than one. Let $p < q < r$ be the prime factors of n. If $p \geq 5$ then $n \geq 5^2\cdot 7\cdot 11 = 1925$. Thus we may assume $p = 3$. By Lemma 4 we know that the power of 3 is either one and some other power is at least two. Or the power of 3 is at least three. If the power of 3 is at least three then $n \geq 3^3 \cdot 5 \cdot 7 = 945$. So we may assume that the power of three is one. Now, if $q \geq 7$ then $n \geq 3 \cdot 7^2 \cdot 11 = 1617$. Thus we may assume $q = 5$, similarly if $r \geq 13$ then $n \geq 3 \cdot 5^2 \cdot 13 = 975$. Thus either $r = 7$ or 11. Therefore the only remaining possibilities less than or equal to 945 are $n_1 = 3 \cdot 5^2 \cdot 7 = 525$, $n_2 = 3 \cdot 5 \cdot 7^2 = 735$, and $n_3 = 3 \cdot 5^2 \cdot 11 = 825$. Direct calculations show that $\sigma(n_1) = (4)(31)(8) = 992 < 2n_1$, $\sigma(n_2) = (4)(6)(57) = 1368 < 2n_2$, and $\sigma(n_3) = (4)(31)(12) = 1488 < 2n_3$. Therefore 945 is the smallest odd abundant number. ■

The proofs of the lemmas are on the next page. But you can try to do them on your own first!

Here are the proofs of the lemmas.

Proof. (of Lemma 2). Since $p \geq 2$, then $p^{a+1} \geq 2p^a$. Thus $p^{a+1}+1 > 2p^a$, adding $p^{a+1}-2p^a-1$ to both sides we get $2p^a(p-1) = 2p^{a+1}-2p^a > p^{a+1}-1$. Thus $2n = 2p^a > (p^{a+1}-1)/(p-1) = \sigma(n)$, that is n is not abundant. ■

Proof. (of Lemma 3). Let $n = pqr$, since n is odd we have that $p \geq 3$, $q \geq 5$, and $r \geq 7$. Thus $1/p \leq 1/3$, $1/q \leq 1/5$, $1/r \leq 1/7$, and

$$\begin{aligned} \frac{\sigma(n)}{n} &= \left(\frac{p+1}{p}\right)\left(\frac{q+1}{q}\right)\left(\frac{r+1}{r}\right) = \\ &= \left(1+\frac{1}{p}\right)\left(1+\frac{1}{q}\right)\left(1+\frac{1}{r}\right) \leq \frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7} < 2. \end{aligned}$$

Thus n is not abundant. ■

Proof. (of Lemma 4). Let $n = 3^2qr$, we have that $q \geq 5, r \geq 7$. Thus $1/q \leq 1/5$, and $1/r \leq 1/7$ and

$$\begin{aligned} \frac{\sigma(n)}{n} &= \left(\frac{1+3+9}{9}\right)\left(\frac{q+1}{q}\right)\left(\frac{r+1}{r}\right) \\ &= \left(\frac{13}{9}\right)\left(1+\frac{1}{q}\right)\left(1+\frac{1}{r}\right) \leq \frac{13}{9}\cdot\frac{6}{5}\cdot\frac{8}{7} < 2. \end{aligned}$$

■

Proof. (of Lemma 5). Since n is odd then we can assume $q > p \geq 3$. We first claim that $pq+2 > 2(p+q)$. If $p=3$ then $pq+2 = 3q+2 \geq 2q+5+2 > 2(3+q)$. If $p \geq 5$ then $pq+2 \geq 5q+2 > 2q+2p = 2(p+q)$. Using the claim we have that

$$p^aq^b(pq+2) > 2p^aq^b(p+q)$$

and since $p^{a+1}+q^{a+1} > 1$ we get

$$p^aq^b(pq+2) + p^{a+1} + q^{a+1} > 2p^aq^b(p+q) + 1$$

which is equivalent to

$$2\left(p^{a+1}q^{b+1} - p^aq^b(p+q) + p^aq^b\right) > p^{a+1}q^{a+1} - p^{a+1} - q^{b+1} + 1$$

that is

$$\begin{aligned} 2p^aq^b(p-1)(q-1) &> \left(p^{a+1}-1\right)\left(q^{b+1}-1\right) \\ 2n &> \frac{\left(p^{a+1}-1\right)\left(q^{b+1}-1\right)}{(p-1)(q-1)} = \sigma(n). \end{aligned}$$

Thus n is not abundant. ■