

Problem of the Week 3, Spring 2006

Solution by organizers. Model this problem with a *graph* G . The vertices (points) of G represent the gnomes' houses and there is an edge (line) joining two vertices if the corresponding gnomes are friends. We color the edges of G . An edge of G is colored yellow if the houses it joins are one blue and one red. Otherwise the edge is colored green.

In a given month, gnome g visits his friends. Suppose for a moment that his house is red. Note that the number of yellow edges from g corresponds to the number of blue houses whose owners are friends with g . Similarly the number of green edges from g corresponds to the number of red houses whose owners are friends with g . If g needs to repaint his house then there are more yellow edges than green edges. Once the house is repainted all edges from g switch colors and now there are more green than yellow edges from g . The same argument holds if g 's house is blue. This means that each month that a house is repainted, the number of yellow edges decreases. Since the number of yellow edges cannot decrease forever, then sooner or later no changes will be needed.