Problem of the Week

Proposed by Bernardo Ábrego and Silvia Fernández. **April 18-25**

We define a recursive sequence of points in the plane as follows: The initial point has coordinates (x_0, y_0) and all other points are obtained from their preceding point according to the formula

$$(x_{n+1}, y_{n+1}) = (x_n + y_n, x_n - y_n).$$

We also know that, after 2005 steps, we obtain the point $(x_{2005}, y_{2005}) = (2^{1003}, 2^{1004})$. Find the coordinates of the initial point.

Solution begins in the next page.

Solution by Marcia Dalla Vechia Ferreira.

Checking the first iterations:

$$(x_0,y_0) = given \tag{1}$$
$$(x_1,y_1) = (x_0+y_0,x_0-y_0) \tag{2}$$
$$(x_2,y_2) = (2x_0,2y_0) \tag{3}$$
$$(x_3,y_3) = (2x_0+2y_0,2x_0-2y_0) \tag{4}$$
$$(x_4,y_4) = (4x_0,4y_0) \tag{5}$$
$$(x_5,y_5) = (4x_0+4y_0,4x_0-4y_0) \tag{6}$$
$$(x_6,y_6) = (8x_0,8y_0) \tag{7}$$

the pattern that appears to emerge is:

$$(x_{2n},y_{2n}) = (2^n x_0,2^n y_0) \tag{8}$$
$$(x_{2n+1},y_{2n+1}) = (2^n x_0+2^n y_0,2^n x_0-2^n y_0) \tag{9}$$

checking that this is the case:
assuming that

$$(x_{2n},y_{2n}) = (2^n x_0,2^n y_0) \tag{10}$$

is true for n. The value for $2n+2$ (the next even number) should be:

$$(x_{2n+2},y_{2n+2}) = (2^{n+1}x_0,2^{n+1}y_0) \tag{11}$$

$$(x_{2n+2},y_{2n+2}) = (x_{2n+1}+y_{2n+1},x_{2n+1}-y_{2n+1}) = \tag{12}$$
$$(x_{2n}+y_{2n}+x_{2n}-y_{2n},x_{2n}+y_{2n}-x_{2n}+y_{2n}) = \tag{13}$$
$$(2x_{2n},2y_{2n}) = (2^{n+1}x_0,2^{n+1}y_0) \tag{14}$$

as expected. The same is true for the odd iteractions.
so, for

$$(x_{2005}, y_{2005}) = (2^{1002}x_0 + 2^{1002}y_0, 2^{1002}x_0 - 2^{1002}y_0) \quad (15)$$

$$(2^{1002}x_0 + 2^{1002}y_0, 2^{1002}x_0 - 2^{1002}y_0) = (2^{1003}, 2^{1004}) \quad (16)$$

$$2^{1002}x_0 + 2^{1002}y_0 = 2^{1003} => x_0 + y_0 = 2 \quad (17)$$

$$2^{1002}x_0 - 2^{1002}y_0 = 2^{1004} => x_0 - y_0 = 4 \quad (18)$$

$$x_0 = 3, y_0 = -1 \quad (19)$$