California State University
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Department of Mathematics

Problem of the Week

Proposed by Bernardo Ábrego and Silvia Fernández. **February 21-28**

Find the smallest positive integer k for which the equation

$$a\left(a^2-k\right)+b\left(b^2-k\right)=2005$$

has a solution with a and b positive integers. Justify your answer.

Solution by Boian Djonov (edited). Solving for k we obtain

$$k=\frac{a^3+b^3}{a+b}-\frac{2005}{a+b}=\frac{(a+b)(a^2-ab+b^2)}{a+b}-\frac{2005}{a+b}=a^2-ab+b^2-\frac{2005}{a+b}.$$

Since we want k, a, and b to be positive integers, then $\frac{2005}{a+b}$ also has to be an integer. So, $a+b$ has to equal to a positive factor of 2005 (2005 = 401*5) which are 1, 5, 401, and 2005.

Case 1: $a+b=1$. Here a and b can not be both positive integers.

Case 2: $a+b=5$. Then, $k=a^2-ab+b^2-401$. In this case k is negative for values of a and b such that $a+b=5$.

Case 3: $a+b=401$. Then,

$$k=a^2-ab+b^2-5=(a+b)^2-3ab-5=401^2-3a(401-a)-5,$$

$$k=160796+3a^2-1203a.$$

Since we want k to have minimum value, we set the derivative of the above expression equal to 0: $6a-1203=0 \Rightarrow a=200.5$. Since a is an integer then the maximum is achieved when $a=200$ or $a=201$. Then $k=160796+3(201)^2-1203(201)=40196$.

Case 4: $a+b=2005$

Similarly to Case 3, I evaluated that the minimum value for k in that case is $k=1005006$.

So, from the three cases it follows that the minimum positive integer value that k can have is: $k=40196$, when $a=201$, $b=200$.