California State University
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Problem of the Week

Proposed by Bernardo Ábrego and Silvia Fernández. **February 14-21**

Let $P(x) = x^3 + c_2x^2 + c_1x + c_0$ be a cubic polynomial with real coefficients and with three different real roots. Let w be the average of the three roots. Prove that $P(x)$ is decreasing in some interval around w.

Solution by Rober Reiner (edited). Let $P(x) = (x - r_1)(x - r_2)(x - r_3)$ where r_1, r_2, and r_3 are the three distinct real roots. Assume $r_1 < r_2 < r_3$. By Rolle's Theorem (or Mean Value Theorem) we can say that between r_1 and r_2 there is a place where the function has a derivative of zero, suppose s_1. Similarly there is a zero value of the derivative s_2 between r_2 and r_3. Now, because $P(x)$ goes to $-\infty$ as x goes to $-\infty$ and it goes to ∞ as x goes to ∞, we can say that $P(x)$ is negative before r_1, positive between r_1 and r_2, negative between r_2 and r_3, and positive after r_3. Then $P(s_1) > 0$, $P(s_2) < 0$ and thus $P(x)$ is decreasing between $s_1$and s_2.

Note that $P'(x) = 3x^2 - 2(r_1 + r_2 + r_3)x + r_1r_2 + r_1r_3 + r_2r_3$. Thus the real roots of the derivative, s_1 and s_2, are located at

$$\frac{r_1 + r_2 + r_3}{3} \pm \frac{\sqrt{4(r_1 + r_2 + r_3)^2 - 12(r_1r_2 + r_1r_3 + r_2r_3)}}{6}.$$

Now, $(r_1 + r_2 + r_3) = w$ so our average of the three roots will be the midpoint of s_1 and s_2 and as we said before $P(x)$ is decreasing in the interval (s_1, s_2).