California State University
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Department of Mathematics

Problem of the Week

Proposed by Bernardo Ábrego and Silvia Fernández. **May 9-16**

A real number x between 0 and 1 is chosen uniformly at random. If $\sqrt{x}$ is written as a decimal in base 10, what is the probability that its first non-zero digit is equal to 3?

Solution by the organizers. Suppose x is a number in the interval [0,1] such that the first non-zero digit of $\sqrt{x}$ is 3. This means that $3 \cdot 10^{-n} \leq \sqrt{x} < 4 \cdot 10^{-n}$ for some $n \geq 1$ integer. Squaring this inequality we get the equivalent inequality

$$9 \cdot 10^{-2n} \leq x < 16 \cdot 10^{-2n}.$$

Thus the set of numbers x in the interval [0,1] satisfying that their first non-zero digit is the union of the intervals $[0.09, 0.16], [0.0009, 0.0016], [0.000009, 0.000016], \ldots, [9 \cdot 10^{-2n}, 16 \cdot 10^{-2n}]$. To find the probability of this set we just need to calculate the sum of the lengths of these intervals. We get

$$\begin{aligned}
& (0.16 - 0.9) + (0.0016 - 0.0009) + \cdots + (16 \cdot 10^{-2n} - 9 \cdot 10^{-2n}) + \cdots \\
= & \sum_{n=1}^{\infty} (16 \cdot 10^{-2n} - 9 \cdot 10^{-2n}) = \sum_{n=1}^{\infty} \left(7 \cdot 10^{-2n}\right) = 7 \sum_{n=1}^{\infty} \left(\frac{1}{100}\right)^n \\
= & \frac{7}{100} \sum_{n=0}^{\infty} \left(\frac{1}{100}\right)^n = \frac{7}{100} \left(\frac{100}{99}\right) = \frac{7}{99} = 0.\overline{07}.
\end{aligned}$$