California State University Northridge

Department of Mathematics

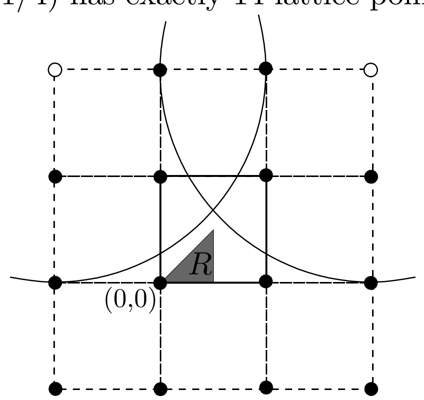
Problem of the Week

Proposed by Bernardo Ábrego and Silvia Fernández. **March 7-14**

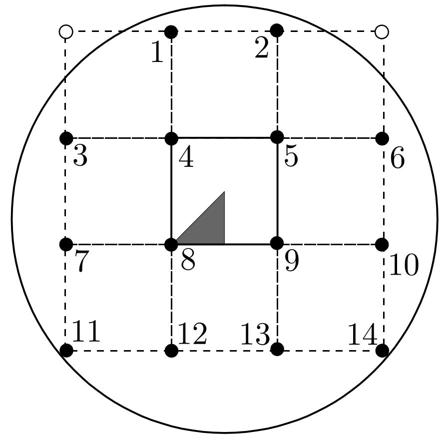
Consider an arbitrary circle of radius 2 in the coordinate plane. What is the largest possible number of *lattice* points inside, but not on, the circle? Justify your answer.

Note: A lattice point is a point whose coordinates are both integer numbers.

Solution (by the organizers). The answer is 14**.** Let C be the circle of radius 2. By translation symmetry we can assume that the center of C is in the unit square with vertices $(0,0),(1,0),(1,1),(0,1)$. Furthermore, by reflection and rotation symmetry of this square, we can assume the center lies inside or on the triangle with vertices $(0,0),(1/2,0),(1/2,1/2)$ (region R in figure (a)). Now, if (n,m) is a lattice point outside of the square given by the points (x,y) with $-1 \leq x \leq 2$, $-1 \leq y \leq 2$,then it clearly has distance at least two from any point in the unit square, hence these lattice points cannot be inside the circle C. This argument already shows that at most 16 lattice points could be inside the circle C. To remove two more points, notice that the circles of radii 2 centered at (-1,2) and (2,2) do not intersect the region R. Hence these two lattice points cannot be inside C. Finally, note that the circle C with center $(1/2,1/4)$ has exactly 14 lattice points in its interior (figure (b)).



(a)



(b)