Problem of the Week. **May 3-10**

Proposed by Bernardo Ábrego and Silvia Fernández.

In an election, two candidates, Carlos and Luis, have in a ballot box 10 and 5 votes respectively. If ballots are randomly drawn and tallied one at a time, what is the probability that Carlos is ahead after each one of the 15 ballots is tallied?

For example, if the ballots are drawn in the order CCLCLLCLLCCCCCC then after the 6th ballot is tallied Carlos and Luis are tied with 3 votes each, so Carlos is not always ahead. However in the order CCLCCLLCLCCCLCC Carlos is always ahead (Carlos-Luis: 1-0, 2-0, 2-1, 3-1, 4-1, 4-2, 4-3, 5-3, 5-4, 6-4, 7-4, 8-4, 8-5, 9-5, 10-5).

Solution by Boian Djonov. I will find first the probability that Carlos is not always ahead in the votes drawn. If Carlos is not always ahead in the votes, this means at some point Luis has to be ahead, or Carlos and Luis have to have the same number of votes drawn at that moment.

However, in order for Luis to be ahead of Carlos, the number of their votes drawn has to become equal before Luis takes the lead. So, Carlos will not be always ahead as soon as the number of his votes drawn becomes equal to the number of Luis' votes drawn. The only exception is when the first drawn vote belongs to Luis (so he is ahead without reaching a tie in the number of Carlos' votes before that). So, I will find the probability of getting a tie in the number of votes of Carlos and Luis.

Some notation: Let $E_{2\text{-}1}$ be the event that, after 3 votes are drawn, Carlos has 2 votes versus 1 vote for Luis, and before that Carlos is always leading in the number of his votes drawn. Similarly we define the events E_{a-b} and we let $P(E_{a\text{-}b})$ refer to the probability of such event.

The possible tie scores are 1-1, 2-2, 3-3, 4-4, and 5-5. Below, I list all possible ways of reaching those ties, such that before the tie occurs Carlos is always leading in the number of his votes drawn. A * next to a score (ex. 2-1*) means that this case has already been considered, so repetition is avoided. I start from the first tie reached, and go backwards toward the initial 0-0 score.

1-1	1-0

$P(E_{1\text{-}1}) = 10/15 \cdot 5/14 = 0.2381.$

2-2	2-1	2-0	1-0

$P(E_{2\text{-}0}) = 10/15 \cdot 9/14 = 0.429.$
$P(E_{2\text{-}1}) = 5/13 \cdot P(E_{2\text{-}0}) = 0.1648.$
$P(E_{2\text{-}2}) = 4/12 \cdot P(E_{2\text{-}1}) = 0.0549.$

			3-0	2-0*
3-3	3-2	3-1		
			2-1*	

$P(E_{3\text{-}0}) = 10/15 \cdot 9/14 \cdot 8/13 = 0.2637.$
$P(E_{3\text{-}1}) = P(E_{3\text{-}0}) \cdot 5/12 + P(E_{2\text{-}1}) \cdot 8/12 = 0.2198.$
$P(E_{3\text{-}2}) = P(E_{3\text{-}1}) \cdot 4/11 = 0.0799.$
$P(E_{3\text{-}3}) = P(E_{3\text{-}2}) \cdot 3/10 = 0.0240.$

				4-0	3-0*
			4-1		
4-4	4-3	4-2		3-1*	
			3-2*		

$P(E_{4\text{-}0}) = P(E_{3\text{-}0}) \cdot 7/12 = 0.1538.$
$P(E_{4\text{-}1}) = P(E_{4\text{-}0}) \cdot 5/11 + P(E_{3\text{-}1}) \cdot 7/11 = 0.2098.$
$P(E_{4\text{-}2}) = P(E_{4\text{-}1}) \cdot 4/10 + P(E_{3\text{-}2}) \cdot 7/10 = 0.1398.$
$P(E_{4\text{-}3}) = P(E_{4\text{-}2}) \cdot 3/9 = 0.0466.$
$P(E_{4\text{-}4}) = P(E_{4\text{-}3}) \cdot 2/8 = 0.0117.$

					5-0	4-0*
				5-1		
			5-2		4-1*	
5-5	5-4	5-3		4-2*		
			4-3*			

$P(E_{5\text{-}0}) = P(E_{4\text{-}0}) \cdot 6/11 = 0.0839.$
$P(E_{5\text{-}1}) = P(E_{5\text{-}0}) \cdot 5/10 + P(E_{4\text{-}1}) \cdot 6/10 = 0.1678.$
$P(E_{5\text{-}2}) = P(E_{5\text{-}1}) \cdot 4/9 + P(E_{4\text{-}2}) \cdot 6/9 = 0.1678.$
$P(E_{5\text{-}3}) = P(E_{5\text{-}2}) \cdot 3/8 + P(E_{4\text{-}3}) \cdot 6/8 = 0.0979.$
$P(E_{5\text{-}4}) = P(E_{5\text{-}3}) \cdot 2/7 = 0.0279.$
$P(E_{5\text{-}5}) = P(E_{5\text{-}4}) \cdot 1/6 = 0.0047.$

The special case when Luis gets the first vote is: $P(E_{0\text{-}1}) = 5/15 = .3333$. So, the probability Carlos is not always ahead in the votes drawn equals:

$$\begin{aligned} & P(E_{0\text{-}1}) + P(E_{1\text{-}1}) + P(E_{2\text{-}2}) + P(E_{3\text{-}3}) + P(E_{4\text{-}4}) + P(E_{5\text{-}5}) \\ = \; & 0.3333 + 0.2381 + 0.0549 + 0.0240 + 0.0117 + 0.0047 = 0.6667. \end{aligned}$$

So, Probability (Carlos is always ahead) = 1 − Probability(Carlos is not always ahead) = 1 − 0.6667 = 0.3333 or about 33.33%.