Problem of the Week. **April 26-May 3**

Proposed by Bernardo Ábrego and Silvia Fernández.

Prove that if nine points are chosen in the interior of a unit square, then three of them form a triangle with area at most 1/8. (Note: If three points are on the same line, we say that the triangle they form has area zero).

Solution by Takumi Saegusa. Let $ABCD$ be a unit square, L, M, N, O be the middle points of the edges AB, BC, CD, and DA, respectively, and P be the intersection of the line AC and BD. Consider partitioning $ABCD$ into 4 squares $ALPO, BMPL, CNPM, DOPN$. The pigeonhole principle shows that at least one of the four squares contains three points since the number of squares is 4 and the number of points is 9. If the point is on the boundary line of squares, I count the point in the one square which consists of the boundary, which doesn't affect my argument.

Without loss of generality, the square $ALPO$ contains three points, say, a_1, a_2, and a_3. If these three points are on the same line, I have nothing to prove. Consider the other case. I claim that there exists a triangle QRS containing the triangle $a_1a_2a_3$ where the points Q, R, S are on some edges of the square $ALPO$. Let Q and R be the intersections of edges of the square $ALPO$ and the line ℓ through a_1 and a_2. Let S be the intersection of the edge of the square $ALPO$ and the line which is perpendicular to the line ℓ, passes through a_3, and such that a_3 is between S and ℓ. Then the triangle QRS satisfies my claim.

Now, I show that the area of the triangle QRS is at most 1/8. If two of the three points Q, R, S are on the same edge, the triangle has area of at most 1/8, since the base formed by these two points on the same edge has length at most 1/2, and the height is at most 1/2. Now, consider the three points Q, R, S are on the different edges. Without loss of generality, suppose Q is on AL, R is on LP, and S is on OP. Let s, t, and u be the length of the segments AQ, LR, and PS respectively. We have $0 \leq s, t, u \leq 1/2$. Then

$$\begin{aligned}
\text{area of } QRS &= (\text{area of } ALPO) - (\text{area of } QLR) - (\text{area of } PRS) - (\text{area of } AQSO) \\
&= \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2} - s\right)t - \frac{1}{2}\left(\frac{1}{2} - t\right)u - \frac{1}{2}\left(s + \left(\frac{1}{2} - u\right)\right)\left(\frac{1}{2}\right) \\
&= \frac{1}{4} - \frac{t}{4} + \frac{st}{2} - \frac{u}{4} + \frac{tu}{2} - \frac{s}{4} - \frac{1}{8} + \frac{u}{4} \\
&= \frac{1}{8} - \left(\frac{1}{2} - u\right)\frac{t}{2} - \left(\frac{1}{2} - t\right)\frac{s}{2} \leq \frac{1}{8}
\end{aligned}$$

since two terms dropped are nonpositive. This completes the proof.