Problem of the Week #5 **April 12-19, 2004**

Proposed by Bernardo Ábrego and Silvia Fernández.

Let $P(x) = x^4 + ax^3 - bx^2 + cx + 1$ be a polynomial with real coefficients. Prove that if $|a + c| < b - 2$ then P has four different real roots (that is, there are four different real values of x for which $P(x) = 0$).

Solution by Takumi Saegusa. Since $|a + c| < b - 2$ then

$$2 - b < a + c < b - 2. \tag{1}$$

We have $P(0) = 1 > 0$ and $P(1) = 1 + a - b + c + 1 = -[(b - 2) - (a + c)] < 0$ by (1). By the intermediate value theorem, there exists a real number d_1 such that $P(d_1) = 0$ and $0 < d_1 < 1$.

We also have $P(-1) = 1 - a - b - c + 1 = -[(a + c) - (2 - b)] < 0$ by (1) and by the intermediate value theorem we have $P(d_2) = 0$, for some d_2 with $-1 < d_2 < 0$.

Since the coefficient of x^4 in $P(x)$ is positive, $P(x)$ goes to infinity as x goes to both the positive and negative infinity. So by the intermediate value theorem, we have $P(d_3) = P(d_4) = 0$ for some d_3 and d_4 with $d_3 > 1$ and $d_4 < -1$. This completes the proof.