Problem of the Week #4 **March 29-April 5, 2004**

Proposed by Bill Watkins.

Guillermo goes to the bank to cash a check. The teller inadvertently switches the cents and the dollars on the check and gives the corresponding amount to Guillermo. After buying a 35 cents newspaper, Guillermo realizes that he has twice as much money as the original value of his check. What was the amount in the check?
Explain how you found your solution and prove there is only one correct answer.

Solution by Barbara J. Falkowski. Guillermo has a check for a dollars and b cents. He is given b dollars and a cents or $b + (1/100)a$ dollars. After buying a newspaper he has $b + (1/100)a - 0.35$ dollars which equals twice what the check should have been worth.

$$b + (1/100)a - 0.35 = 2(a + (1/100)b).$$

Rearranging the terms we get

$$(98b - 35)/100 = 199a/100.$$

For a and b to be easily swapped they must have at most two digits each. This gives the following constraints,

$$0 \leq a \leq 99,\ 0 \leq b \leq 99.$$

For b to be less than or equal to 99 the limits on a must be redefined as

$$b = (199a + 35)/98 \leq 99.$$

From this condition we find that $0 \leq a \leq 48$.

Since b must be an integer $(199a+35)/98$ must be an integer and since $98 = 2 \times 7 \times 7$, $199a+35$ must be evenly divisible by 2 and 7 twice. For $199a + 35$ to be divisible by 2, a must be odd. Since 35 is divisible by 7 and 199 is not, a must be divisible by 7. The numbers between 0 and 48 that are odd and evenly divisible by 7 are $7, 21$, and 35. Now it is a matter of putting them in the equation and seeing if either gives an integer value for b.

$$\begin{aligned} b_1 &= (199(7) + 35)/98 \approx 14.57 \\ b_2 &= (199(21) + 35)/98 = 43 \\ b_3 &= (199(35) + 35)/98 \approx 71.42. \end{aligned}$$

Only $a = 21$ gives an integer value for b, namely 43. So the check was for 21 dollars and 43 cents.