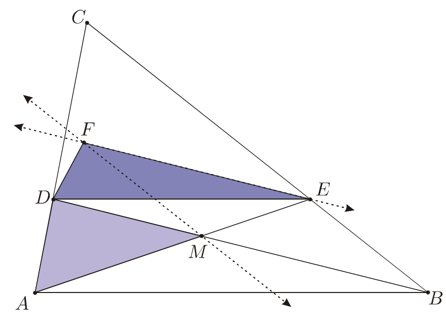
Problem of the Week #3 **March 22-29, 2004**

Proposed by Bernardo Ábrego and Silvia Fernández.

Let ABC be an arbitrary triangle. Let D and E be points on $\overline{AC}$ and $\overline{BC}$ respectively such that $\overline{DE}$ is parallel to $\overline{AB}$. Let M be the intersection of $\overline{BD}$ and $\overline{AE}$. Let F be the intersection of the line through M parallel to $\overline{BC}$, and the line through E parallel to $\overline{BD}$. Prove that the triangles DEF and AMD have the same area.



Solution by Takumi Saegusa. We have a theorem that says that triangles on equal bases and between the same parallels are equal in area. (a)

Since the triangles DEF and EFM have the line segment EF in common and since the line segments DM and FE are parallel, the areas of the triangles DEF and EFM are the same by (a). (1)

Since the triangles EFM and BFM have the line segment FM in common and since the line segments BE and MF are parallel, the areas of the triangles EFM and BFM are the same by (a). (2)

Since the triangles BEM and BFM have the line segment BM in common and since the line segment BM and EF are parallel, the areas of the triangles BEM and BFM are the same. (3)

Since the triangles ABE and ABD have the line segment AB in common and since the line segments AB and DE are parallel, the areas of the triangles ABE and ABD are the same. (4)

Then

$$
\begin{aligned}
& \text{(the area of the triangle } AMD) \\
= \ & \text{(the area of the triangle } ABD) - \text{(the area of the triangle } ABM) \\
= \ & \text{(the area of the triangle } ABE) - \text{(the area of the triangle } ABM) \text{ by (4)} \\
= \ & \text{(the area of the triangle } BEM) \\
= \ & \text{(the area of the triangle } BFM) \text{ by (3)} \\
= \ & \text{(the area of the triangle } EFM) \text{ by (2)} \\
= \ & \text{(the area of the triangle } DEF) \text{ by (1).} \qquad \text{Q.E.D.}
\end{aligned}
$$