Problem of the Week. **May 3-10**

Proposed by Bernardo Ábrego and Silvia Fernández.

In an election, two candidates, Carlos and Luis, have in a ballot box 10 and 5 votes respectively. If ballots are randomly drawn and tallied one at a time, what is the probability that Carlos is ahead after each one of the 15 ballots is tallied?

For example, if the ballots are drawn in the order CCLCLLCLLCCCCCC then after the 6th ballot is tallied Carlos and Luis are tied with 3 votes each, so Carlos is not always ahead. However in the order CCLCCLLCLCCCLCC Carlos is always ahead (Carlos-Luis: 1-0, 2-0, 2-1, 3-1, 4-1, 4-2, 4-3, 5-3, 5-4, 6-4, 7-4, 8-4, 8-5, 9-5, 10-5).

First solution (by organizers). We first compute the probability that Carlos is **not always** ahead. Note that, since Carlos will definitely be ahead once the 15 votes have been tallied, the probability that Carlos is not always ahead is equal to the probability that Carlos and Luis are tied at some moment. We prove that

$$P\left(\text{there is a tie, first vote C}\right) = P\left(\text{there is a tie, first vote L}\right).$$

To do this we give a one-to-one correspondence between the set of sequences starting with C containing a tie and the set of sequences starting with L containing a tie. Given a sequence starting with C containing a tie, switch the Cs and Ls from the beginning of the sequence to the first tied position (the rest of the sequence is not changed). For example, in the sequence CCLCLLCLLCCCCCC the first tie occurs at the sixth position so the sequence starting with L associated with it is LLCLCCCLLCCCCCC.

Every sequence of 10 Cs and 5 Ls that starts with L satisfies that at some point Luis and Carlos will be tied (since Luis starts winning but Carlos will always take over). Then

$$P\left(\text{there is a tie, first vote L}\right) = P\left(\text{first vote L}\right) = \frac{5}{15} = \frac{1}{3}.$$

Therefore

$$\begin{aligned} P\left(\text{there is a tie}\right) &= P\left(\text{there is a tie, first vote C}\right) + P\left(\text{there is a tie, first vote L}\right) \\ &= 2P\left(\text{first vote L}\right) = \frac{2}{3}, \end{aligned}$$

and

$$P\left(\text{C always ahead}\right) = 1 - P\left(\text{there is a tie}\right) = 1 - \frac{2}{3} = \frac{1}{3}.$$

Second solution (by organizers). Consider the 10×5 grid shown in the picture. Note that the number of ways the votes can be tallied in such a way that Carlos is always ahead, is equal to the number of paths on the grid joining the points $(0,0)$ and $(10,5)$ that do not touch the line $y = x$ under the restriction that the paths only allowed one unit movements to the right or upward. The path RRURRURRRRUURRU (where R=right and U=up) corresponds to the sequence CCLCCLCCCCLLCCL. To count these paths we indicate the number of ways to get from $(0,0)$ to any of the nodes (a,b) with $a \geq b$, i.e. the nodes on the right side of the line $y = x$.

Note that the number of ways to get to $(a+1, b+1)$ can be computed by adding the number of ways to get to $(a+1, b)$ plus the number of ways to get to $(a, b+1)$. So, for example, once we know that the number of ways to get to $(7,2)$ is 20 and the number of ways to get to $(6,3)$ is 28 then we get that the number of ways to get to $(7,3)$ is $20+28=48$. We show all these numbers in the picture (note that it looks like a Pascal triangle with vertex at $(0,0)$. Therefore the number of ways to tally the votes in such a way that Carlos is always ahead is 1001. Since the total number of ways to tally the votes is $\binom{15}{5} = 3003$ (choose the 5 places for the Ls in the sequnce of length 15) then the probability we are looking for is $\frac{1001}{3003} = \frac{1}{3}$.