Problem of the Week. **April 12-19**

Proposed by Bernardo Ábrego and Silvia Fernández.

Let $P(x) = x^4 + ax^3 - bx^2 + cx + 1$ be a polynomial with real coefficients. Prove that if $|a+c| < b-2$ then P has four different real roots (that is, there are four different real values of x for which $P(x) = 0$).

Solution (by organizers). Even though this solution is not as simple as the winner's solution, it requires no Calculus and it may be used for some other related problems.

First, any 4th degree polynomial with real coefficients can be factored into two quadratic polynomials, so there are real numbers A_1, B_1, A_2, and B_2 such that

$$P(x) = \left(x^2 + A_1 x + B_1\right)\left(x^2 + A_2 x + B_2\right). \tag{1}$$

We know that a quadratic polynomial has two different real roots if and only if the discriminant is positive. So let us assume, by contradiction, that $P(x)$ does not have four different roots because the discriminant of $\left(x^2 + A_1 x + B_1\right)$ is non positive. i.e., we assume that

$$(A_1)^2 - 4B_1 \leq 0 \text{ or } (A_1)^2 \leq 4B_1. \tag{2}$$

Now, if we expand the product in (1) we get

$$P(x) = x^4 + (A_1 + A_2)\,x^3 + (B_1 + B_2 + A_1 A_2)\,x^2 + (A_1 B_2 + A_2 B_1)\,x + B_1 B_2,$$

thus we must have that

$$\begin{aligned} a &= A_1 + A_2, \\ -b &= B_1 + B_2 + A_1 A_2, \\ c &= A_1 B_2 + A_2 B_1, \text{ and} \\ 1 &= B_1 B_2. \end{aligned}$$

Thus $B_2 = 1/B_1$ which together with (2) implies that $B_1 > 0$. Then

$$\begin{aligned} a + c &= A_1 + A_2 + \frac{A_1}{B_1} + A_2 B_1 = A_1\left(\frac{B_1 + 1}{B_1}\right) + A_2\,(B_1 + 1) = (B_1 + 1)\left(\frac{A_1 + A_2 B_1}{B_1}\right) \\ b - 2 &= -B_1 - \frac{1}{B_1} - A_1 A_2 - 2 = -\left(\frac{B_1^2 + 2B_1 + 1}{B_1}\right) - A_1 A_2 = -\frac{(B_1 + 1)^2}{B_1} - A_1 A_2. \end{aligned}$$

Now, by assumption $|a+c| < b-2$, thus

$$\frac{(B_1 + 1)}{B_1}\,|A_1 + A_2 B_1| < -\frac{(B_1 + 1)^2}{B_1} - A_1 A_2,$$

and since $B_1 > 0$ we can multiply by B_1, also for simplicity we let $k = A_2 B_1$. Then

$$(B_1 + 1)\,|A_1 + k| < -(B_1 + 1)^2 - A_1 k.$$

Completing the square and simplifying,

$$\begin{aligned}
\left(B_1+1+\frac{|A_1+k|}{2}\right)^2 &< \frac{|A_1+k|^2}{4}-A_1k\\
\left(B_1+1+\frac{|A_1+k|}{2}\right)^2 &< \frac{A_1^2+2A_1k+k^2-4A_1k}{4}\\
\left(B_1+1+\frac{|A_1+k|}{2}\right)^2 &< \left(\frac{|A_1-k|}{2}\right)^2.
\end{aligned}$$

Then, since both terms inside the parenthesis are positive, we can take square roots on both sides and then

$$B_1+1<\frac{|A_1-k|}{2}-\frac{|A_1+k|}{2}.$$

Now we use $A_1^2/4 \le B_1$ from (2) to get rid of B_1.We get

$$A_1^2+4<2\,|A_1-k|-2\,|A_1+k|\,. \tag{3}$$

Finally, if A_1-k and A_1+k have different signs then

$$2\,|A_1-k|-2\,|A_1+k|=\pm 4A_1\le 4\,|A_1|\le A_1^2+4,$$

where the last inequality is true since $0\le(|A_1|-2)^2=A_1^2-4\,|A_1|+4$. This is a contradiction to (3).

On the other hand, if A_1-k and A_1+k have the same sign then this means that $|k|\le|A_1|$ and then

$$2\,|A_1-k|-2\,|A_1+k|=\pm 4k\le 4\,|k|\le 4\,|A_1|\le A_1^2+4,$$

which is again a contradiction to (3). This completes the proof.