Problem of the Week #4 **March 29-April 5, 2004**

Proposed by Bill Watkins.

Guillermo goes to the bank to cash a check. The teller inadvertently switches the cents and the dollars on the check and gives the corresponding amount to Guillermo. After buying a 35 cents newspaper, Guillermo realizes that he has twice as much money as the original value of his check. What was the amount in the check?
Explain how you found your solution and prove there is only one correct answer.

Solution (by organizers). Suppose that the original amount of the check is x dollars and y cents, that is, the check was made for $100x + y$ cents. The problem states that

$$\underbrace{100y + x}_{\text{switch cents and dollars}} - \underbrace{35}_{\text{cost of newspaper}} = \underbrace{2\,(100x + y)}_{\text{twice original amount}} .$$

Reducing this equation we get

$$98y - 199x = 35, \tag{1}$$

where x and y are integers and, since y is the number of cents, $0 \le y \le 99$. This is a *Diophantine equation.* Any Diophantine equation like (1) with no restrictions has either no solutions or infinitely many. Our equation (without the restriction $0 \le y \le 99$) has infinitely many solutions because $\gcd(98, 199) = 1$ divides 35. To find all solutions to the equation we first need to find one solution. You can guess or find the solution by trial and error. However, the *Euclidean algorithm* is a technique you can use to find one solution (if there are any) to any Diophantine equation like (1).

Here is how the algorithm works. We start with the pair $(199, 98)$. In each row we divide the first number in the pair by the second. This gives a quotient (in this case 2) and a reminder (in this case 3). Then we can write $199 = 2\,(98) + 3$. Our next pair consists of the second number in the last pair and the reminder obtained in the last division. We repeat this procedure until the reminder is 0. Putting this together we get

$$\begin{aligned} 199 &= 2\,(98) + 3 \\ 98 &= 32\,(3) + 2 \\ 3 &= 1\,(2) + 1 \\ 2 &= 1\,(2) + 0. \end{aligned}$$

It is known that the last nonzero remainder is equal to the greatest common divisor of the first pair of numbers. In this case $\gcd(199, 98) = 1$.

Now we use the previous computations to write $\gcd(199, 98) = 1$ as a linear combination of 199 and 98 (that is we are finding an integer solution to $199a + 98b = 1$. We isolate the reminders in each of the rows, except the last one.

$$\begin{aligned} 199 - 2\,(98) &= 3 \\ 98 - 32\,(3) &= 2 \\ 3 - 1\,(2) &= 1. \end{aligned}$$

Now we combine this three rows starting from the bottom.

$$\begin{aligned}
3-1\,(2) &= 1 \quad \text{(3rd row)} \\
3-1\,(98-32\,(3)) &= 1 \quad \text{(use 2nd row to replace 2 by } 98-32\,(3)\text{)} \\
33\,(3)-98 &= 1 \quad \text{(put together the 3s and 98s)} \\
33\,(199-2\,(98))-98 &= 1 \quad \text{(use 1st row to replace 3 by } 199-2\,(98)\text{)} \\
33\,(199)-67\,(98) &= 1 \quad \text{(put together the 98s and 199s).} \qquad (2)
\end{aligned}$$

We have a solution to $199a + 98b = 1$ (namely $a = 33, b = -67$) but we need a solution to $98y - 199x = 35$. So we multiply (2) by 35.

$$\begin{aligned}
35 \cdot [33\,(199) - 67\,(98)] &= 35 \cdot 1 \\
(35 \cdot 33)\,(199) - (35 \cdot 67)\,(98) &= 35 \\
(1155)\,(199) - (2345)\,(98) &= 35 \\
98\,(-2345) - 199\,(-1155) &= 35. \qquad (3)
\end{aligned}$$

So one solution to the equation $98y - 199x = 35$ is $y = -2345, x = -1155$. Note that (3) is equivalent to

$$98\,(-2345 + 199k) - 199\,(-1155 + 98k) = 35,$$

since $98\,(199k) - 199\,(98k) = 0$. It is known that **all solutions** to the equation $98y - 199x = 35$ have the form

$$y = -2345 + 199k, x = -1155 + 98k,$$

where k is any integer number.

In our problem we have the restriction $0 \leq y \leq 99$. So

$$\begin{aligned}
0 &\leq -2345 + 199k \leq 99 \\
11.784 &\approx \frac{2345}{199} \leq k \leq \frac{99 + 2345}{199} \approx: 12.281
\end{aligned}$$

Since k must be an integer, its only possible value is 12. This gives

$$\begin{aligned}
y &= -2345 + 199k = -2345 + 199\,(12) = 43 \\
x &= -1155 + 98k = -1155 + 98\,(12) = 21,
\end{aligned}$$

and so the unique solution to the problem is \$21.43.