Problem of the Week
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We need to randomly pick 3 points on the unit circle and find the expected value of area of the resulting triangle is. This is a geometric probability problem. As is always the case in probability problems, the expected value will be number of favorable outcomes divided by total number of outcomes. In this case, we are looking at a geometric probability, so it is the area of the favorable outcome divided by total area which will turn out to be $\frac{1}{2\pi^2}$.

Using the symmetry of the circle and WOLOG, we can fix one point to start with, so lets's fix $a = (0, 1)$. Next we randomly pick 2 other points, b and c on the circumference of the unit circle. We call the central angles between a and b, θ_1 , and between b and c, θ_2. Once again, because of symmetry of the circle, we can consider $\theta_1 \in [0, \pi]$and $\theta_2 \in [0, 2\pi)$.

Now we note that the area of the triangle formed by a, b and c is $A = 2r^2 \sin(\frac{1}{2}\ \theta_1) \sin(\frac{1}{2}\ \theta_2) \sin[\frac{1}{2}\ (\theta_1 - \theta_2)]$. Now since we are on the unit cirle, $r = 1$ and the area of the triangle triangle formed by a, b, and c is:

$A = \left|2 \sin(\frac{1}{2}\theta_1) \sin(\frac{1}{2}\theta_2) \sin[\frac{1}{2}(\theta_1 - \theta_2)]\right|$

Next, we need to integrate A to find the total area of the triangles and then divide that value by the total possible area.

So we have :

$$2\int_0^{\pi}\int_0^{2\pi} \left|\sin(\tfrac{\theta_1}{2}) \sin(\tfrac{\theta_2}{2}) \sin[\tfrac{(\theta_1-\theta_2)}{2}]\right| d\theta_2 d\theta_1$$

divided by $\int_0^{\pi}\int_0^{2\pi} d\theta_2 d\theta_1$

and $\int_0^{\pi}\int_0^{2\pi} d\theta_2 d\theta_1 = 2\pi^2$

So our expected value is $\frac{1}{\pi^2}\int_0^{\pi}\int_0^{2\pi} \left|\sin(\frac{\theta_1}{2}) \sin(\frac{\theta_2}{2}) \sin[\frac{(\theta_1-\theta_2)}{2}]\right| d\theta_2 d\theta_1$

Now, using a computer algebra system, and a bit of work to evaluate the integrals, we find that our expected area is

$\frac{1}{\pi^2}[\frac{3\pi}{2}] = \frac{3}{2\pi}$