Problem of the Week 12, Fall 2008

Determine the expected value of the area of a triangle whose vertices are chosen uniformly and independently at random on a circumference of radius 1.

Solution by organizers. The expected value is $\frac{3}{2\pi}$. To prove this, we model the random space as the triples (α, β, γ), where each of the entries represents the central angle of each of the sides of the triangle. Also, each of the entries is chosen independently an uniformly at random from the interval $[0, 2\pi]$. For a given triple (α, β, γ), the area of the corresponding triangle equals

$$A(\alpha, \beta, \gamma) = \frac{1}{2}\left\| \begin{array}{ccc} \cos\alpha & \sin\alpha & 1 \\ \cos\beta & \sin\beta & 1 \\ \cos\gamma & \sin\gamma & 1 \end{array} \right\|.$$

(The absolute value of the determinant.) Thus we have that

$$\mathbb{E}(A) = \frac{1}{(2\pi)^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} A(\alpha, \beta, \gamma) d\alpha\, d\beta\, d\gamma.$$

Because $A(\alpha, \beta, \gamma) = A((\alpha - \gamma) \bmod 2\pi, (\beta - \gamma) \bmod 2\pi, 0)$, it follows that

$$\begin{aligned}
\mathbb{E}(A) &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} A(\alpha, \beta, 0) d\alpha\, d\beta \\
&= \frac{1}{2\,(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \left\| \begin{array}{ccc} \cos\alpha & \sin\alpha & 1 \\ \cos\beta & \sin\beta & 1 \\ 1 & 0 & 1 \end{array} \right\| d\alpha\, d\beta \\
&= \frac{1}{2\,(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \left|\sin\alpha - \sin\beta + \cos\alpha\sin\beta - \cos\beta\sin\alpha\right| d\alpha\, d\beta \\
&= \frac{1}{2\,(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \left|\sin\alpha - \sin\beta + \sin(\beta - \alpha)\right| d\alpha\, d\beta.
\end{aligned}$$

Because the expression

$$\left| \begin{array}{ccc} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ 1 & 0 & 1 \end{array} \right|$$

is positive if and only if the triangle with vertices $(x_1, y_1), (x_2, y_2), (1, 0)$ (in that order) is oriented in counterclockwise direction, it follows that $\alpha \le \beta$ if and only if $\sin\alpha - \sin\beta + \sin(\beta - \alpha) \ge 0$.

Thus

$$
\begin{aligned}
\mathbb{E}(A) &= \frac{1}{8\pi^2}\int_0^{2\pi}\left(\int_0^{\beta}(\sin\alpha-\sin\beta+\sin(\beta-\alpha))\,d\alpha+\int_{\beta}^{2\pi}(-\sin\alpha+\sin\beta-\sin(\beta-\alpha))\,d\alpha\right)d\beta \\
&= \frac{1}{8\pi^2}\int_0^{2\pi}\left(\int_0^{\beta}(\sin\alpha-\sin\beta+\sin(\beta-\alpha))\,d\alpha+\int_{\beta}^{2\pi}(-\sin\alpha+\sin\beta-\sin(\beta-\alpha))\,d\alpha\right)d\beta \\
&= \frac{1}{8\pi^2}\int_0^{2\pi}(\cos(\alpha-\beta)-\alpha\sin\beta-\cos\alpha)\Big|_{\alpha=0}^{\alpha=\beta}+(-\cos(\alpha-\beta)+\alpha\sin\beta+\cos\alpha)\Big|_{\alpha=\beta}^{\alpha=2\pi}\,d\beta \\
&= \frac{1}{8\pi^2}\int_0^{2\pi}((2-\beta\sin\beta-2\cos\beta)+(2\pi\sin\beta-2\cos\beta-\beta\sin\beta+2))\,d\beta \\
&= \frac{1}{8\pi^2}\int_0^{2\pi}(2\pi\sin\beta-4\cos\beta-2\beta\sin\beta+4)\,d\beta \\
&= \frac{1}{8\pi^2}\,(4\beta-6\sin\beta-2\pi\cos\beta+2\beta\cos\beta)\Big|_{\beta=0}^{\beta=2\pi}=\frac{12\pi}{8\pi^2}=\frac{3}{2\pi}.
\end{aligned}
$$