Problem of the Week 11, Fall 2008

Determine all the positive integers that are equal to the square of their number of positive divisors.

Solution by organizers. Clearly $n = 1$ and $n = 9$ work. We prove that these are the only possibilities. Assume $n > 1$. Let $d(n)$ denote the number of positive divisors of a positive integer n. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the unique (except for order) prime factorization of the integer n, then every positive divisor b has the form $p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$ with $0 \leq \beta_i \leq \alpha_i$, $1 \leq i \leq k$. Thus the number of positive divisors is equal to $d(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1)$. The given conditions of the problem imply that

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} = (d(n))^2 = (\alpha_1 + 1)^2 (\alpha_2 + 1)^2 \cdots (\alpha_k + 1)^2.$$

Thus n is a perfect square, that is, all the exponents α_i are even, say $\alpha_i = 2A_i$. Our new identity is

$$n = p_1^{2A_1} p_2^{2A_2} \cdots p_k^{2A_k} = (d(n))^2 = (2A_1 + 1)^2 (2A_2 + 1)^2 \cdots (2A_k + 1)^2.$$

Because the number on the right is clearly odd, it follows that $p_i \geq 3$ for $1 \leq i \leq k$.

We prove the following by induction on A.

Claim 1 *If $p \geq 3$ and $A \geq 1$, then $p^A \geq (2A + 1)$, with equality only for $p = 3$ and $A = 1$.*

Proof. If $A = 1$ and $p \geq 3$, then $p^A = p \geq 3 = 2A + 1$, with equality only when $p = 3$. Assume $p^A \geq 2A + 1$; then by induction

$$p^{A+1} = p \cdot p^A \geq p\,(2A + 1) > 2A + p \geq 2A + 3.$$

Because the inequality $p^{A+1} > 2A + 3$, then equality holds only when $(p, A) = (3, 1)$. ■

Back to our problem, if $k \geq 2$ then $p^{A_i} \geq (2A_i + 1)$ for $1 \leq i \leq k$ and at least one inequality is strict. Thus

$$n = \left(p_1^{A_1} p_2^{A_2} \cdots p_k^{A_k}\right)^2 > ((2A_1 + 1)(2A_2 + 1) \cdots (2A_k + 1))^2 = (d(n))^2.$$

The same conclusion holds if $k = 1$ and $p_1 \geq 5$. The only remaining possibilities are $k = 1$ and $p_1 = 3$, that is $n = 9$.